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S-GEAR: A FRAMEWORK FOR PROOF INVESTIGATION AND RECONSTRUCTION IN REAL ANALYSIS

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Abstrak

Mahasiswa Tadris Matematika IAIN Parepare menghadapi tantangan besar dalam memahami dan menyusun pembuktian Analisis Real karena keterbatasan dalam mengenali model-model pembuktian, prinsip deduktif, serta ketentuan formal yang membentuk argumen matematis. Ketiadaan mata kuliah pendukung seperti Logika Matematika semakin membatasi kemampuan mereka membaca struktur bukti dan menafsirkan simbol penting, termasuk definisi yang melibatkan ε–δ. Penelitian ini mengembangkan strategi S-GeaR (Starting Point, Goals, Idea, References) sebagai kerangka investigasi bukti untuk memandu mahasiswa menganalisis dan merekonstruksi pembuktian teorema secara sistematis. Dengan model pengembangan ADDIE, S-GeaR dirancang, divalidasi, dan diimplementasikan melalui investigasi teorema, diskusi kolaboratif, serta lembar kerja pembuktian. Data diperoleh melalui tes, observasi kelas, dan refleksi mahasiswa. Hasil kuantitatif menunjukkan peningkatan nilai dari 58,2 menjadi 76,4 ($p = 0{,}012$; $N - Gain = 0{,}43$). Melaui penggunaan lembar kerja S-GeaR mahasiswa memperlihatkan perkembangan bertahap dalam aktivitas pembuktian: mengenali jenis metode bukti, menggali ide, hingga menyusun bukti formal lengkap. Temuan kualitatif juga menunjukkan bahwa melalui instruksi prosedural menurunkan beban kognitif mahasiswa melalui pemetaan komponen bukti, serta mendukung penggalian ide pembuktian. Temuan ini menunjukkan bahwa S-GeaR berperan penting dalam memperkuat struktur penalaran mahasiswa dalam pembuktian teorema.

Kata kunci: *Strategi S-GeaR, Pembuktian Teorema Analisis Real, Rekonstruksi Bukti, Penalaran Matematis*

Abstract

Students in the Mathematics Education Department at IAIN Parepare face substantial challenges in understanding and constructing proofs in Real Analysis due to limited familiarity with proof models, deductive principles, and the formal structures that underpin mathematical arguments. The absence of supporting courses such as Mathematical Logic further restricts their ability to read proof structures and interpret essential symbolic forms, including definitions involving ε–δ. This study developed the S-GeaR (Starting Point, Goals, Idea, References) strategy as a proof-investigation framework designed to guide students in systematically analyzing and reconstructing theorem proofs. Using the ADDIE development model, S-GeaR was designed, validated, and implemented through theorem investigation, collaborative discussions, and structured proof worksheets. Data were collected through tests, classroom observations, and student reflections. Quantitative results show an increase in scores from 58.2 to 76.4 ($p = 0.012$; $N - Gain = 0.43$). Through the use of S-GeaR worksheets, students demonstrated gradual development in proof-related activities: recognizing proof methods, generating ideas and composing a complete formal proof. Qualitative findings also indicate that the procedural guidance of S-GeaR reduces cognitive load through component mapping and supports the emergence of proof ideas. These results underscore the important role of S-GeaR in strengthening students' reasoning structure in theorems proofs.

Keywords: Mathematical Reasoning; Proof Reconstruction; Real Analysis Theorem Proof; S-GeaR Strategy

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INTRODUCTION

Real Analysis is widely acknowledged as one of the most conceptually demanding courses in undergraduate mathematics due to its abstract foundations and its rigorous expectations for formal proof construction. These demands are particularly evident among students in the Mathematics Education Department at IAIN Parepare. Observational data and diagnostic evaluations collected from 2020 to 2023 consistently reveal fundamental difficulties: students misread theorem assumptions, misinterpret quantified statements, fail to differentiate the antecedent from the consequent in logical implications, and struggle to convert symbolic expressions into coherent deductive arguments. Collectively, these issues indicate a persistent reliance on procedural habits inherited from calculus rather than the deductive reasoning essential for engaging with formal proofs.

These difficulties are compounded by curricular conditions. Unlike programs at major universities, the Tadris Matematika curriculum at IAIN Parepare lacks foundational courses such as Mathematical Logic or Introduction to Proofs, leaving students without a conceptual framework for reading or constructing rigorous proofs. Prior learning, focused on computational algorithms, creates an epistemic gap when students encounter ε–δ definitions, chains of logical implication, and proof techniques such as direct proof, contradiction, or contraposition.

These difficulties are further intensified by the characteristics of the primary course text, *Introduction to Real Analysis* (Robert & Donald, 2011). Although widely used, the textbook presents proofs in a dense, abstract, and compressed style, stated directly as complete deductive paragraphs without explicit structural decomposition. This makes it difficult for novice learners to trace logic, identify assumptions, extract intermediate goals, and discern the central ideas, providing the final proof form but not the investigative process behind it, and leaving a pedagogical gap for students learning proof construction.

These observations are consistent with national research indicating that students struggle with logical symbolism and constructing valid deductive arguments (Lestari, 2015; Siregar, 2015). as well as studies documenting errors in convergence proofs (Helma et al., 2018; Kristianto & Saputro, 2019). Globally, similar challenges emerge when students transition from computational to deductive mathematics (David & Zazkis, 2020). These converging findings highlight the need for instructional approaches that provide explicit guidance in investigating and constructing proofs.

Existing pedagogical approaches such as the Analysis Boot Camp (Seager, 2020), theorem-structure frameworks (Selden, 2013), concept mapping (Khotimah et al., 2019), historical approaches (Bressoud, 2020), and the REACT model (Khusna, 2020) offer valuable insights but tend to address isolated aspects of proof learning. They do not provide a systematic framework that guides students through the full investigative process of analyzing, decomposing, and reconstructing proofs. This indicates a pedagogical gap that warrants the development of a more comprehensive instructional strategy.

The S-GeaR (Starting Point, Goals, Idea, References) framework was developed to address challenges in proof comprehension. Grounded in constructivist learning and cognitive load theories, S-GeaR decomposes proofs into four analytical components that guide students in tracing assumptions, identifying theorem goals, articulating key ideas, and selecting relevant references. By structuring proofs into investigative stages, it facilitates logical flow, reduces cognitive load, and enables systematic reconstruction of formal arguments.

This study aims to develop a proof-investigation strategy using the S-GeaR framework, design a worksheet instrument for theorem analysis, assess improvements in students' proof comprehension, and evaluate the pedagogical potential of S-GeaR in enhancing Real Analysis reasoning at IAIN Parepare, bridging the gap between conceptual understanding and formal proof construction identified in prior research (David & Zazkis, 2020; Kristianto & Saputro, 2019; Selden, 2013).

METHOD

Research Design

This study employed a developmental research design to construct and evaluate the S-GeaR strategy for enhancing proof comprehension in Real Analysis. The iterative approach combined expert validation, small-group testing, classroom implementation, and mixed-methods evaluation, ensuring rigor and replicability in line with contemporary standards in mathematics education research.

Participants and Ethical Considerations

The study involved 17 undergraduate students enrolled in the 2024 Real Analysis course in the Mathematics Education Department at IAIN Parepare. Ethical clearance was granted by the Institutional Ethics Review Board. Participation was voluntary, informed consent was obtained from all participants, and anonymity was upheld throughout the data collection and analysis processes.

Development of the S-GeaR Framework

The conceptual foundation of S-GeaR was established through systematic analysis of classical proofs in Bartle's *Introduction to Real Analysis*, focusing on deductive flow, inferential transitions, and implicit reasoning often compressed in textbooks. Analytic decomposition of selected theorems identified four essential components, Starting Point (S), Goals (G), Idea (ea), and References (R), which form the structural pillars of the S-GeaR model. Insights from diagnostic interviews and documented student difficulties, particularly with quantified statements, logical implication chains, and strategy selection, further refined this conceptual framework.

Design and Validation of Instructional Materials

Based on the conceptual framework, instructional tools were developed, including S-GeaR worksheets, theorem-structure sheets, model proof decompositions, and instructor guides. These materials were validated by two Real Analysis experts and one instructional design specialist, using criteria of clarity, coherence, logical precision, cognitive-load appropriateness, and pedagogical usability, with revisions made accordingly to align with learning objectives.

Prototype Development and Small-Group Trial

A prototype of the S-GeaR module was piloted with ten students to assess usability, readability, and clarity of instructions. Feedback and observed errors guided refinements, ensuring the final version was cognitively manageable and pedagogically appropriate.

Implementation of S-GeaR in Classroom Instruction

The refined S-GeaR strategy was implemented in the Real Analysis course, where students used worksheets to identify theorem structures, articulate assumptions and conclusions, explore proof ideas, select relevant references, and reconstruct formal proofs following the S-GeaR sequence. Instruction combined guided practice, small-group collaboration, instructor clarification, and class discussions, emphasizing analytical proof construction while reducing cognitive load and making the investigative process explicit.

Data Collection Procedures

Data were collected from pre- and post-tests, classroom observations, S-GeaR worksheets, and student reflections. Tests assessed theorem interpretation, logical structuring, and proof construction, while observations and worksheet analyses captured reasoning behaviors and proof reconstruction. Reflections provided insights into cognitive load and the effectiveness of the S-GeaR framework.

Data Analysis Techniques

Quantitative data were analyzed using descriptive statistics and paired samples t-tests to compare pre- and post-test results. Normalized gain (N-gain) scores were calculated to quantify improvement, and *Wilcoxon signed-rank test* with 95% confidence intervals were reported to assess practical significance. Qualitative data from reflections, observations, and worksheets were analyzed through thematic coding.

RESULT AND DISCUSSION

Analysis Stage

An initial analysis revealed persistent challenges for students in understanding and constructing Real Analysis proofs. Observations of Mathematics Education students at IAIN Parepare (2021–2024) showed difficulties in recognizing logical structures, interpreting symbols, and handling implicative propositions, quantifiers, and specialized limit notation. Limited familiarity with proof symbols and resolution conventions further hindered students' ability to follow, interpret, and reconstruct sequential reasoning steps.

These local findings align with previous research. Lestari (2015) reported students' difficulties in reading proofs and constructing coherent arguments, while Siregar Siregar (2015) highlighted weak reasoning skills and limited introductory resources. Sobarningsih et al. (2019) found that inadequate understanding of theorem structures often causes confusion in proof construction, and Sucipto and Mauliddin (2017) noted challenges in applying definitions and deductive reasoning. Furthermore, Sherbert and Bartle (2011) observed that common textbooks present overly formal proofs with minimal guidance on patterns and linguistic coherence.

Building on these insights, a needs analysis identified three key instructional supports for improving students' proof comprehension. First, a procedural framework is needed to simplify logical structures and symbols, guiding students in identifying proof components and understanding implications and quantifiers. Second, beginner-friendly resources should model proof patterns and use connective sentences to reduce the dominance of formal symbols. Third, supportive instructional media, such as interactive modules for step-by-step proof exercises, are necessary to foster active engagement, drawing on Seager's (2020) foundational proof training model. Collectively, these findings provided the rationale for developing the S-GeaR (Starting Point, Goals, Idea, References) strategy, designed to guide students in systematically investigating and reconstructing proofs in Real Analysis.

Design Results

Design Philosophy and Framework

The S-GeaR strategy was designed from a constructivist perspective, emphasizing that mathematical understanding develops through structured exploration and active knowledge reconstruction. In proof-based learning, students must internalize the logical connections between premises and conclusions, yet traditional Real Analysis instruction often prioritizes formal symbolic procedures over cognitive reasoning. The S-GeaR framework was thus developed as a pedagogical scaffold, a cognitive map guiding learners to analyze, plan, and systematically reconstruct proofs, aimed at bridging the gap between intuitive reasoning and formal proof construction.

Textbook Exploration as Design Input

A textbook exploration was conducted to examine how proof presentation in major Real Analysis references informs pedagogical design. Bartle and Sherbert (2011) *Introduction to Real Analysis* remains one of the most respected sources, offering a highly structured format with definitions, examples, theorems, and lemmas, accompanied by an appendix on logic and proof techniques. While thorough in formal rigor, this presentation assumes that readers already possess symbolic logic competence, providing minimal conceptual scaffolding for beginners.

Comparative review of Cunningham's *Real Analysis with Proof Strategies* (2021), Grinberg's *The Real Analysis Lifesaver* (2017), and Stoll's *Introduction to Real Analysis* (2021) revealed similar tendencies. Cunningham provides organized descriptions of proof techniques; Grinberg introduces active "fill-in-the-blank" exercises to complete proofs; and Stoll demonstrates strong coherence between sentences and symbolic structures. Despite these strengths, most textbooks remain formal and top-down, rarely deconstructing proofs into digestible cognitive steps. These observations indicated that existing references successfully convey formal mathematical rigor but fall short in guiding learners through the reasoning process. Consequently, the S-GeaR strategy was designed to reinterpret textbook proofs into a structured investigative approach that enables students to explore *how* and *why* each proof step functions within a logical framework.

According to Suandito (2017), a formal proof consists of a sequence of logically coherent arguments derived from accepted premises and axioms. In real analysis, proof serves not only as a tool for verification but also as a means for developing logical thinking and mathematical communication. Bartle and Sherbert (2011) classify proofs in real analysis into three primary forms: direct proof, indirect proof by contraposition, and proof by contradiction. Other essential types include proof by counterexample and proof by induction (Grinberg, 2017). Each form of proof requires learners to identify assumptions, determine objectives, establish logical relationships, and employ previously proven results.

These evidentiary structures became the logical foundation of S-GeaR's design. The four components were intentionally aligned with the mental steps students undertake during proof construction: identifying what is known, defining what is to be shown, developing conceptual strategies, and linking to relevant theorems. This mapping ensures that the strategy adheres to authentic proof reasoning while remaining pedagogically accessible for novice learners.

Design Implementation and S-GeaR Components

Based on analyses of student difficulties, proof logic, and textbook structures, the S-GeaR strategy was developed as a four-step framework to support investigative learning in proof-based mathematics. Implemented via structured worksheets and instructor guides, Starting Point prompts students to restate given information symbolically and verbally, Goals translates the desired conclusion into precise logical terms, Idea provides guiding questions for strategy selection, and References directs identification of theorems justifying each inferential step. This framework shifts proof learning from passive imitation to active investigation, enabling students to reconstruct proofs with clarity and awareness of underlying reasoning, while promoting both procedural accuracy and conceptual depth.

Table 1. S-GeaR Descriptions

Component of S-GeaR	Concise Definition	Function in Proof Construction
Starting Point (S)	The given assumptions, premises, or quantified statements that initiate the proof.	Establishes the logical foundation and separates antecedent from consequent in implicative statements.
Goals (G)	The statement to be proven, whether one-directional or bi-directional.	Directs the deductive pathway from hypothesis to conclusion and clarifies the proof's objective.
Idea (ea)	The key conceptual insight that connects the Starting Point to the Goal.	Activates the central reasoning step, guides strategy selection, and reduces cognitive load.
References (R)	Definitions, theorems, lemmas, or properties invoked to justify proof steps.	Ensures coherence by grounding the argument in established mathematical results.

Design Frameworks

To operationalize the conceptual model of the S-GeaR framework, the design process mapped the cognitive structure of mathematical proof onto four interrelated phases: *textbook analysis, identification of proof elements, evidentiary surgery, and evidentiary reconstruction*. These phases represent a continuum from the formal presentation of proofs in standard texts to their deconstruction and reconstruction through the S-GeaR lens.

Table 2 illustrates the pedagogical intent of the S-GeaR strategy: translating formal proofs into structured cognitive activities. By linking each phase, Starting Point, Goals, Idea, and References, to evidentiary reasoning, the framework provides instructors and learners with a systematic guide for proof reconstruction. This synthesis bridges symbolic formalism and conceptual understanding, reinforcing the strategy's constructivist and cognitive learning foundations, and informed the development and validation of instructional materials in the study's next phase.

Table 2. Worksheet of S-GeaR in Proof Investigation and Reconstruction

The Proof in the Textbook	S-GeaR Identification
Theorem 3.1.4 The statement in the theorem*:* *General assumptions, Hypotheses, consequences.* **Proof in Textbooks: .....** **Type of proof: ....**	**StartingPoint (S):** Initial assumptions: - Hypothesis: **Goals:** Consequence/Conclusion: **Idea:** Bring up a new object, Relate it to the definitions, properties or theorems that have been proven, Operating hypothesis, Leading to conclusions **References:** Key definitions, Previous theorem that has been proved, Properties, axioms
Evidentiary Surgery	Evidentiary Reconstruction

Hypothesis Decomposition: Description Assumptions that refer to properties, definitions or postulates. **Goals Decomposition:** Decomposition of conclusions based on references (definitions, properties, theorems that have been proved. **Idea:** A processing direction that appears suddenly to reconcile given hypotheses with intended goals, either explicitly or implicitly. **References:** **Proof Construction:** Sentence construction that combines the decomposition of hypothesis, idea and reference towards the intended conclusion. **Proof Symbolization: Q.ED Proven** ☐	**It will be proved that** *goals* means it **will be proved that** Decomposition of goals. Because of the assumption, it **holds:** Since the *assumption* **then holds:** And based on the hypothesis, then **View , ...** Combination of hypothesis decomposition, ideas and references. By knowing that ..... **this means meeting the *goals*** **Conclusions met** **Q.E.D Proven** ☐

Development Results

Expert Validation Process

After the design phase, the S-GeaR prototype, including its conceptual map, instructional module, and student worksheet, underwent expert validation to ensure theoretical consistency, pedagogical relevance, and content accuracy. Two domain experts and two instructional design specialists evaluated the materials using structured sheets with quantitative ratings and qualitative feedback, assessing clarity of theoretical foundation, internal consistency of components, alignment with proof-based learning objectives, linguistic accuracy, and classroom feasibility. The prototype received an average rating of 87.7% (very valid), with feedback highlighting its effectiveness in translating proof logic into stepwise reasoning and suggesting simplification of terminology in the Idea section and additional guiding examples for References.

Prototype Refinement

Revisions from expert feedback focused on three areas: the module layout was enhanced with visual icons for each S-GeaR component to support cognitive recall; instructional text was simplified to reduce linguistic complexity while preserving logical precision; and the student worksheet was augmented with an "evidence tracker" for recording hypotheses, sub-ideas, and references. A second validation confirmed the revised materials were practical for classroom use and "ready for limited trial," with only minor adjustments needed for contextual examples in local Real Analysis topics.

Small Group Trial

A limited trial with 10 Mathematics Education students in Real Analysis assessed the practicality and initial usability of the S-GeaR materials. Students worked in groups to analyze textbook theorems using the S-GeaR worksheet, while observations and semi-structured interviews captured responses and challenges. Results showed clearer identification of hypotheses and goals, initial difficulty distinguishing Idea from References, and increased confidence in reconstructing proofs. The layout promoted active discussion and reduced reliance on rote memorization, leading to minor wording and example adjustments before broader implementation.

Summary of Development Phase

The development results indicate that the S-GeaR framework received strong expert validation and demonstrated initial classroom practicality. Revisions yielded a pedagogically robust tool for teaching proof construction, validating the model's internal logic and providing a foundation for subsequent implementation and evaluation.

Implementation Stage

Following expert validation and prototype refinement, the S-GeaR strategy was implemented in a limited classroom setting during the 2023/2024 Real Analysis course at IAIN Parepare. Seventeen students, organized into four collaborative groups, worked on theorem investigations using S-GeaR worksheets, with the lecturer facilitating and observing. Data were collected through pre- and post-tests, student reflections, and classroom observations to evaluate practicality, engagement, and cognitive improvement.

In March-June 2024, the S-Gear Strategy was formally implemented in the 6th Semester real analysis course. The strategy involved enhancing the understanding of proofs through practice questions and testing. An example of this is the proof investigation concerning the order properties of real numbers as Theorem 2.1.7.c about "rules of inequalities" (Robert & Donald, 2011). Evaluation results from the S-Gear implementation in the Real Analysis Course yielded the following outcomes.

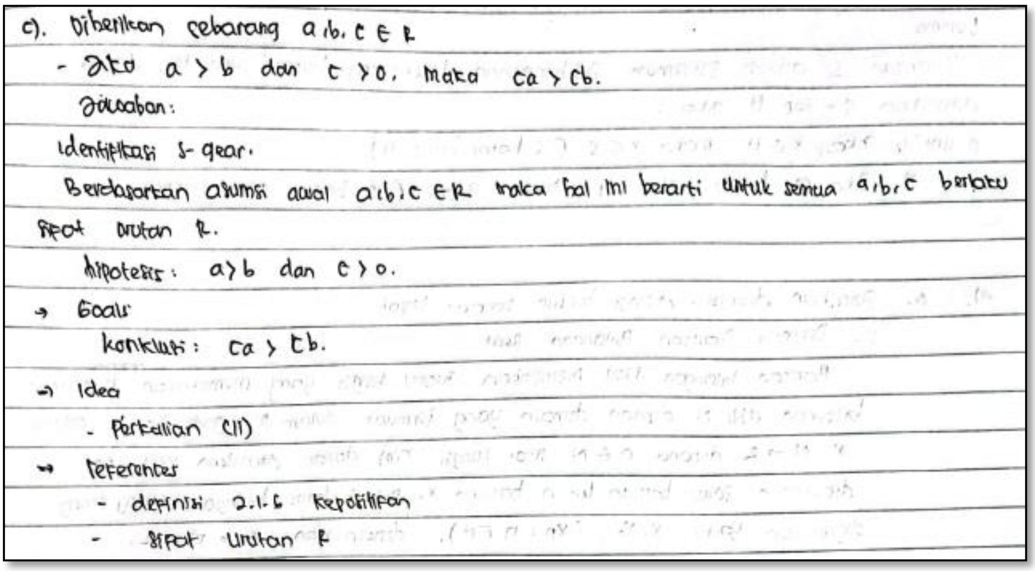


Figure 1. Student's Worksheet Response For Proving an Order Property Using The Definition of Positivity: S-Gear Identification Stage

English Translation of the Student's Proof Work (Raw Extraction Version)

Problem : If $a > b$and $c > 0$, then $ca > cb$.

Based on the initial assumption $a, b, c \in \mathbb{R}$, this means $a - b \in \mathbb{P}$(the set of positive numbers).

Starting Point :

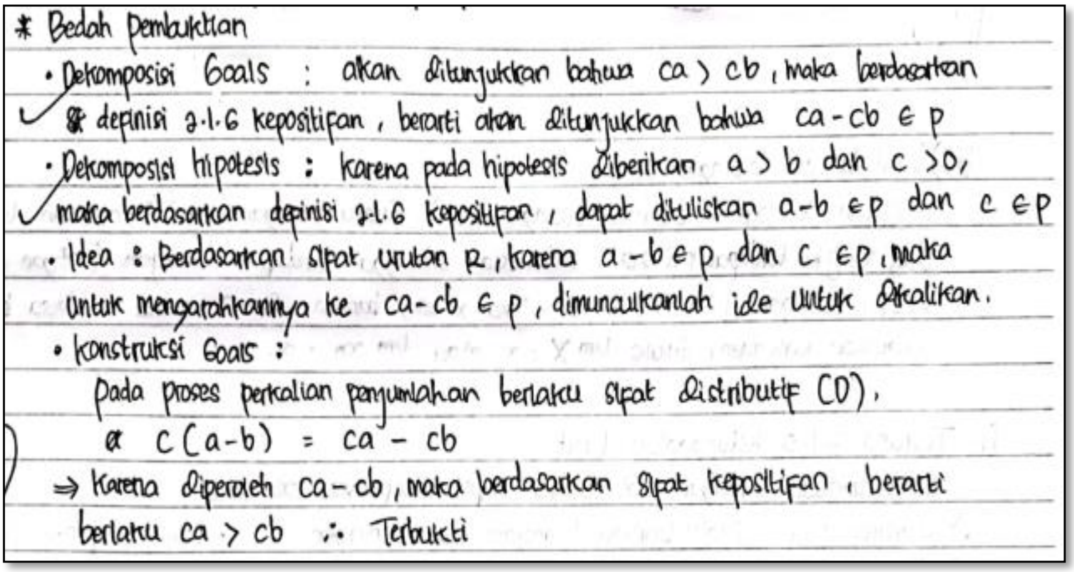
Hypotheses: $a > b, c > 0$

Goals : To prove that $ca > cb$

Idea : Multiply $a - b$and $c$directly.

References :

1. Definition of positivity
2. Order properties and multiplication



* Bedah Pembuktian

• Dekomposisi Goals : akan ditunjukkan bahwa ca > cb, maka berdasarkan definisi 2.1.6 kepositifan, berarti akan ditunjukkan bahwa ca - cb ∈ P

• Dekomposisi hipotesis : karena pada hipotesis diberikan a > b dan c > 0, maka berdasarkan definisi 2.1.6 kepositifan, dapat dituliskan a-b ∈ P dan c ∈ P

• Idea : Berdasarkan sifat urutan R, karena a - b ∈ P dan c ∈ P, maka untuk mengarahkannya ke ca - cb ∈ P, dimunculkanlah ide untuk dikalikan.

• Konstruksi Goals :

Pada proses perkalian penjumlahan berlaku sifat distributif (D).

c(a-b) = ca - cb

⇒ Karena diperoleh ca - cb, maka berdasarkan sifat kepositifan, berarti berlaku ca > cb ∴ Terbukti

Figure 2. Continuation of The Student's Worksheet Response: S-Gear-Based Decomposition of The Proof of an Order Property

Proof Decomposition (Student Answer in English Version)

Decomposition of Goals : We want to prove that $ca > cb$, so based on the definition of positivity, it must be shown that $ca - cb \in \mathbb{P}$.

Decomposition of Hypotheses: Since the hypothesis states that $a > b$and $c > 0$, then based on the definition of positivity, we can write $a - b \in \mathbb{P}$and $c \in \mathbb{P}$.

Idea : Based on the order property of $\mathbb{R}$, because $a - b \in \mathbb{P}$and $c \in \mathbb{P}$, we multiply them to obtain : $c(a - b) \in \mathbb{P}$. This motivates the idea to construct the expression: $ca - cb = c(a - b)$.

Goal Construction : In the multiplication and subtraction process, the distributive property holds: $c(a - b) = ca - cb$. Since $ca - cb \in \mathbb{P}$, we conclude $ca > cb$. Therefore, the statement is proven.

To provide a clearer illustration of how students apply the S-GeaR strategy in constructing mathematical proofs, the following table presents an analysis of a student's worksheet response to an order-property proof task. This analysis identifies each S-GeaR component as it appears in the student's work and compares it with the mathematically accurate structured version. The table highlights the student's reasoning patterns, the correctness of each component, and areas that require conceptual refinement.

Table 3. S-GeaR Analysis of Student's Proof Work

S-GeaR Component	Identified From Student's Work	Structured and Corrected Version
Starting Point (S)	"Hypotheses: $a > b$, $c > 0$."	Given $a, b, c \in \mathbb{R}$with $a > b$and $c > 0$. From the definition of order, $a - b > 0$and $c > 0$.
Goals (G)	"To show $ca > cb$. It must be shown that $ca - cb \in P$."	The goal is to prove that $ca > cb$. Equivalently, show that the difference $ca - cb = c(a - b)$is a positive number. If $ca - cb > 0$, then $ca > cb$.
Idea (ea)	Student idea: Multiply $a - b$ and c directly. Since both $a - b > 0$and $c > 0$, the product $c(a - b)$is also positive.	Multiply $a - b$and $c$directly. Since both are positive, their product is positive: $c(a - b) > 0$. Then rewrite the expression as $ca - cb = c(a - b)$.
References (R)	Student lists: 1. Properties of algebra in $\mathbb{R}$ 2. Order and multiplication 3. Definition of positivity	1. Definition: $x > y \Leftrightarrow x - y \in P$(positivity) 2. Closure of positive numbers under multiplication: if $x > 0$and $y > 0$, then $xy > 0$ 3. Distributive law: $ca - cb = c(a - b)$
Reconstruction of Proof	Student writes an informal sequence of steps relating to subtracting and manipulating inequalities.	*Formal Reconstruction:* 1. From $a > b$, we get $a - b > 0$. 2. From $c > 0$, we also have $c > 0$. 3. The product of two positive numbers is positive, so $c(a - b) > 0$. 4. By the distributive property: $ca - cb = c(a - b)$. 5. Hence $ca - cb > 0$, which implies $ca > cb$. ✓ Proven.

The Instructional Use of S-GeaR in Proof Dissection

During implementation, the lecturer introduced the Real Analysis course by reinforcing prior calculus concepts and highlighting links to advanced analysis. Content was presented systematically through definitions, examples, theorems, lemmas, and proofs, emphasizing logical continuity. A collaborative learning environment with study groups and shared tasks fostered active engagement, while the lecturer explained the S-GeaR framework and guided its procedural application. The following presents a concise application of the S-GeaR framework to the proof of the uniqueness of limits as Theorem 4.1.5 , structured according to the S-GeaR worksheet in Table 2.

Table 4. Worksheet of S-GeaR Implementation

The Proof in The Textbook	S-Gear Identification
Theorem Statement : If $f: A \to \mathbb{R}$ and c is a cluster point of A, then f can have only one limit at c. **Proof in Textbooks:** Suppose that two real numbers L and L' satisfy definition 4.1.4 about the limit definition at c. Then for eny $\varepsilon > 0$, there exists a number $\delta_1(\varepsilon/2) > 0$ such that whenever $x \in A$ and $0 < \mid x - c \mid < \delta_1(\varepsilon/2)$, then $\mid f(x) - L \mid < \frac{\varepsilon}{2}$. Similarly, since L' is also a limit of f at c, there exists $\delta_2(\varepsilon/2) > 0$ such that for every $x \in A$ with $0 < \mid x - c \mid < \delta_2(\varepsilon/2)$, we have $\mid f(x) - L' \mid < \varepsilon/2$. Define $\delta = \min\{\delta_1(\varepsilon/2), \delta_2(\varepsilon/2)\}$. If $x \in A$ and $0 < \mid x - c \mid < \delta$, then both inequalities hold. Hence, by the Triangle Inequality, $$\mid L - L' \mid \le \mid L - f(x) \mid + \mid f(x) - L' \mid < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$ Since ε is arbitrary, this forces $\mid L - L' \mid = 0$, and therefore $L = L'$. **Type of proof: Directly proof**	**StartingPoint (S):** **Initial Assumption:** $f: A \to \mathbb{R}$. **Hypothesis:** c is a cluster point of A. **Goals (Conclusion to be proved):** • The function f has a *unique* limit at the point c. • Formally, if both L and L' satisfy the limit definition at c, then we must show $L = L'$. **Idea (Key Reasoning Idea):** • Assume two possible limits of f at c, denoted by L and L'. • Choose two δ consist pf : $\delta_L(\varepsilon/2)$ and $\delta_{L'}(\varepsilon/2)$ **References:** **Definition 4.1.4:** ε–δ definition of the limit of a function. **Definition of a cluster point:** Guarantees the existence of $x \neq c$ arbitrarily close to c.
Proof Investigations	Formal Proof Reconstruction
Hypothesis Decomposition: Since the hypothesis states that c is a cluster point of A, the definition of a cluster point guarantees that for every $\delta > 0$, there exists an $x \in A, x \neq c$, such that $\mid x - c \mid < \delta$. **Goals Decomposition:** "It will be shown that f can have only one limit at c based on the definition of the limit." For any $\varepsilon > 0$, $\exists \delta > 0$, $x \in A$, $0 < \|x - c\| < \delta$, $\|f(x) - L\| < \varepsilon$. "This means that whenever f has a limit, that limit must be unique; in other words, the value L is uniquely determined."	**It will be proved that** f can have only one limit, it means it **will be proved that** $\forall \varepsilon > 0$, $\exists \delta > 0$, $x \in A$, $0 < \|x - c\| < \delta$, $\|f(x) - L\| < \varepsilon$. Suppose L and L' are both limits of f at c. Let $\varepsilon > 0$. Then for $\varepsilon/2$ there exist δ_1 and δ_2 satisfying the limit conditions for L and L'. Let $$\delta = \min(\delta_1, \delta_2).$$ Since c is a cluster point, choose $x \neq c$ with $0 < \mid x - c \mid < \delta$. Then:

The Proof in The Textbook	S-Gear Identification
Idea Decomposition: • Because there are two possible limits of f at c, denoted by L and L'. • It means, we have to Choose a common $\delta = \min\,(\delta_L(\varepsilon/2), \delta_{L'}(\varepsilon/2))$. • For an arbitrary $\varepsilon > 0$, each limit individually admits a corresponding δ-value (specifically for $\varepsilon/2$). • Use a point x sufficiently close to c and apply the Triangle Inequality to relate L, $f(x)$, and L'. • This will show that $\mid L - L' \mid < \varepsilon$, and since ε is arbitrary, conclude that $L = L'$. **Goals Construction:** Next, let δ be the minimum of the two values $\delta(\varepsilon/2)$ and $\delta'(\varepsilon/2)$. If we choose any $x \in A$ such that $0 < \mid x - c \mid < \delta$, then we obtain $\mid L - L' \mid < \varepsilon$ for all $\varepsilon > 0$. Since $\varepsilon > 0$ is arbitrary, it follows that $\mid L - L' \mid = 0 \quad \Leftrightarrow \quad L - L' = 0,$ and therefore, $L = L'.$ **Proof Symbolization:** **Q.ED** ■ **Proven**	$\mid L - L' \mid \le \mid L - f(x) \mid + \mid f(x) - L' \mid < \varepsilon/2 + \varepsilon/2 = \varepsilon.$ Since ε is arbitrary, $\mid L - L' \mid = 0$, hence $L = L'$. Q.E.D. ■ **Proven**

Learning Process and Student Engagement

During implementation, students actively identified Starting Points and Goals of theorems, and group discussions showed heightened attention to the logical flow of proofs. They increasingly verbalized relationships between hypotheses and conclusions, reflecting a shift from rote memorization to reasoning-based understanding. Early sessions revealed hesitation in the Idea phase, but instructor facilitation helped most groups articulate structured insights, demonstrating growing conceptual coherence. Observations indicated that the S-GeaR worksheet promoted peer explanation, reduced reliance on the lecturer, and encouraged explicit reasoning, supporting collaborative meaning-making and enhanced metacognitive awareness in proof construction.

To document how the S-GeaR framework supported student reasoning during classroom implementation, Table 3 outlines the sequence of student activities aligned with each S-GeaR stage and the corresponding proofing capabilities developed. This mapping illustrates how students progressed from identifying statements and hypotheses to reconstructing formal proofs, revealing the cognitive and procedural growth facilitated by the S-GeaR strategy.

Table 5 . Students Activites and Proofing Capability

Student Activities	S-GeaR Stages	Proofing Capability
Students discern the type of the specified statement. The statement may be classified as implicative, bi-implicative, or equivalently.	Preparation	Recognize types of statements
Students categorize proofs into direct, indirect by contradiction, or indirect by contraposition.	Preparation	Choosing the type of proof
Students condense the theorem into an implication.	Preparation	Decomposition
Students identify their Starting Point and Goals (S-G)	*Starting Point* and *Goals*	Recognize Assumptions, hypotheses
Students identify the references needed based on the identification of starting points and goals.	References	Decomposition of Assumptions, *Hypotheses* and *Goals*
Students recognize intuitively derived *ideas (ea)* and properties. This section categorizes ideas into those that are overtly generated and those that illustrate the trajectory of the proof.	*Idea*	Choosing an idea
Students identify the types of linking sentences associated with hypotheses and those that precede conclusions.	*Connector*	*Idea Construction*
Students reconstruct the proof process by identifying Starting Points, Goals, Ideas, and References, incorporating relevant connecting sentences.	*Evidence Reconstruction*	Compiling Formal Evidence

Pre- and Post-Test Evaluation

To assess preliminary learning outcomes, a short proof comprehension test was administered before and after implementation. The average pre-test score was 58.2, while the post-test average increased to 76.4, with an N-Gain score of 0.43 (*medium category*). A nonparametric *Wilcoxon signed-rank test* was conducted to examine changes, yielding $p = 0.012 < (0.05)$, indicating a statistically significant improvement in students' proof comprehension. The greatest improvements occurred in five proofing-capability domains:

Table 6. Proofing Capability Improvements

Proofing Capability Aspect	Description	Improvement (%)
Recognizing types of statements	Ability to classify statements as implicative, equivalent, or bi-conditional.	32%
Identifying Starting Points and Goals	Skill in articulating given assumptions and stating the precise target of the proof.	38%
Generating appropriate proof ideas	Ability to select or derive key ideas that bridge assumptions to conclusions.	29%
Selecting relevant references	Accuracy in choosing definitions, theorems, or properties needed to justify steps.	41%
Constructing complete and coherent formal proofs	Competence in assembling a fully structured and logically consistent proof.	35%

Student Reflections and Instructor Notes

Qualitative reflections revealed that students found S-GeaR helpful for "seeing the flow of proof" and "knowing what to do first." Several noted that writing down *Starting Points* clarified what assumptions were being used. One student commented:

> *"Before S-GeaR, I memorized the steps of proofs. Now I understand what each line means and how it connects to the theorem."*

The instructor's observation journal indicated a positive classroom dynamic, marked by collaborative reasoning, fewer procedural errors, and improved use of mathematical language. Minor issues remained, particularly in differentiating *Idea* from *References* when proofs involved multiple intermediate results.

Evaluation Summary

Overall, the implementation confirmed that the S-GeaR strategy is practical, pedagogically sound, and cognitively supportive for students learning Real Analysis proofs. Evaluation results demonstrated that student engagement and reasoning articulation increase. The second, Moderate improvement in proof comprehension $(N - Gain = 0.43)$. The third, statistically significant learning gains $(p < 0.05)$, and Positive qualitative perceptions from students and instructors. These findings validate the *usability* and *pedagogical relevance* of S-GeaR and justify its inclusion as a structured tool for investigative proof learning. Future studies may extend the implementation to larger samples and diverse topics to assess broader applicability.

Student Trial of S-GeaR Strategy

The S-GeaR has been integrated into the Real Analysis course at the Mathematics Education study program, IAIN Parepare during the 2024/2025 even semester. This exemplifies a proof investigation of the Limit Uniques Theorem conducted by students through collaborative assignments. Interview findings indicate that students perceive the S-GeaR as beneficial.

In addition, qualitative responses from students were also evaluated concerning the practicality and operational impact of S-GeaR. The results of interviews conducted with two students representing high, medium, and low ability categories are presented here. Interviews with three students one exhibiting high proficiency, another with moderate capability, and a third demonstrating lower aptitude yielded insights. The findings indicate that students of high ability feel substantially supported by the S-GeaR framework; however, they express a degree of trepidation when tasked with proving novel theorems. Conversely, they report that existing theorems are more readily understood and assimilated. For students with lower proficiency, the S-GeaR strategy facilitates the identification of its elements but remains complex to implement in practice.

The challenges and errors encountered by students in the realm of proof construction pertaining to real analysis are closely linked to insufficient comprehension of proof techniques, stemming from both classroom instruction and independent study. Moreover, the manner in which material is presented in standard references for real analysis often lacks clarity, being predominantly abstract and laden with specialized symbols accompanied by minimal elucidation. Consequently, there is a pressing need for initiatives aimed at alleviating students' challenges in grasping the logical flow of proofs.

Discussion

Interpretation of Findings

The findings of this study indicate that the S-GeaR framework supports students' gradual transition from intuitive understanding to formal proof construction. The significant increase in post-test results ($p = 0.012 < 0.05$) and the medium $N - Gain$ (0.43) demonstrate that students not only improved their accuracy in proof solving but also developed procedural fluency. This aligns with Selden's (2013) concept of *proof comprehension*, which emphasizes identifying hypotheses and goals as the foundation of logical reasoning.

Students' reflections confirmed that S-GeaR reduced cognitive barriers by providing structured stages that simplified complex proof processes. From a constructivist perspective, this structure allowed learners to build their own understanding through guided discovery. Simultaneously, the segmentation of tasks within S-GeaR resonates with Cognitive Load Theory (De La Fuente & Altermatt, 2019; Sweller, 1994), as it divides the proof process into manageable units, decreasing extraneous cognitive demand and allowing focus on intrinsic reasoning.

The S-GeaR framework also demonstrates value as a metacognitive tool. Students reported greater confidence in organizing arguments, checking logical coherence, and verbalizing mathematical relationships using connecting statements such as "based on the definition" or "by theorem." These behaviors reflect internalization of logical syntax and argumentation key outcomes of higher-order proof learning.

Theoretical and Pedagogical Implications

Theoretically, S-GeaR advances proof pedagogy by linking conceptual reasoning with procedural understanding, providing a scaffold that supports abstract reasoning without compromising rigor. Pedagogically, its systematic stages offer a framework for teaching proofs across mathematical topics, particularly in bridging foundational courses such as Calculus with advanced analysis.

Limitations and Future Directions

Despite its promising outcomes, this study was limited by a small participant group and short implementation duration. Further research should expand the population, include a control group, and employ mixed-method designs to evaluate long-term retention and transfer of reasoning skills. Additionally, digital or interactive adaptations of S-GeaR may enhance accessibility and sustain student engagement.

CONCLUSION

The findings indicate that the S-GeaR strategy substantially enhances students' proof comprehension in Real Analysis, improving their ability to identify logical structures, articulate Starting Points and Goals, generate proof ideas, select relevant references, and reconstruct coherent formal proofs. Pre–post tests, worksheet analyses, and qualitative reflections show that S-GeaR makes implicit proof structures explicit and reduces the cognitive load of abstract, narrative-style proofs.

This study has limitations, including a small sample size, a one-semester implementation, and challenges with the Idea component, compounded by the absence of foundational courses such as Mathematical Logic in the IAIN Parepare curriculum. These constraints point to future research opportunities, including application of S-GeaR in other proof-intensive courses, comparative studies with alternative instructional approaches, and longitudinal evaluations. Practically, S-GeaR can be implemented via worksheets, guided identification of assumptions and conclusions, and careful selection of prior results, functioning both as a proof-investigation framework and a pedagogical tool to support students' transition from procedural learning to higher-level deductive reasoning.

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