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CRITICAL THINKING AND GEOMETRIC IMAGINATION DEVELOPMENT THROUGH ANALYTIC GEOMETRY: A STUDY ON CIRCLES AND TANGENT LINES

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Abstrak

Pembelajaran geometri analitik menjadi jembatan kognitif antara representasi visual dan aljabar, terutama pada topik lingkaran dan garis singgung. Penelitian deskriptif kualitatif ini bertujuan memetakan proses berpikir siswa SMA ketika menyelesaikan soal nonrutin geometri analitik serta mengidentifikasi bagaimana berpikir kritis dan imajinasi geometri muncul melalui koordinasi representasi visual dan simbolik. Partisipan penelitian adalah 25 siswa kelas XI IPA SMA Santa Maria Cirebon yang dipilih secara purposive karena telah mempelajari materi prasyarat dan topik penelitian serta menunjukkan keragaman kemampuan. Data dikumpulkan melalui empat soal uraian nonrutin, dokumentasi jawaban tertulis, observasi pembelajaran, dan wawancara semi-terstruktur terhadap enam subjek terpilih. Analisis data mengikuti model interaktif Miles, Huberman, dan Saldaña, dengan pengodean berbasis indikator berpikir kritis dari Facione dan Ennis serta indikator representasi visual dari Duval. Hasil menunjukkan bahwa 8 dari 25 siswa (32%) mampu menyelesaikan seluruh soal secara benar dan lengkap. Siswa yang berhasil menunjukkan alur berpikir visualisasi awal, penalaran aljabar, dan evaluasi melalui verifikasi visual. Sebaliknya, siswa yang kurang berhasil cenderung menggunakan rumus secara prosedural tanpa mengaitkan makna geometri dengan bentuk aljabarnya. Temuan ini menunjukkan bahwa pada pembelajaran konvensional, berpikir kritis dan imajinasi geometri dapat muncul ketika siswa dibiasakan membuat sketsa, menerjemahkan kondisi geometri ke persamaan, serta memeriksa kembali hasil secara visual. Implikasi praktisnya, guru perlu memberi *scaffolding* eksplisit pada tahap sketsa awal dan verifikasi visual; namun, karena penelitian ini terbatas pada satu kelas dan bersifat deskriptif kualitatif, temuan harus dibaca sebagai pemetaan proses berpikir kontekstual, bukan klaim efektivitas yang dapat digeneralisasikan secara luas.

Kata kunci: Geometri Analitika; Berpikir Kritis; Imajinasi Geometri; Lingkaran dan Garis Singgung; Visualisasi.

Abstract

Analytic geometry functions as a cognitive bridge between visual and algebraic representations, particularly in the topics of circles and tangent lines. This qualitative descriptive study aims to map high school students' thinking processes when solving non-routine analytic geometry problems and to identify how critical thinking and geometric imagination emerge through the coordination of visual and symbolic representations. The participants were 25 Grade XI Natural Science students at SMA Santa Maria Cirebon, selected purposively because they had studied the prerequisite material and the target

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topic and represented varied levels of ability. Data were collected through four non-routine essay problems, students' written work, classroom observation, and semi-structured interviews with six selected subjects. Data analysis followed the interactive model of Miles, Huberman, and Saldaña, supported by theory-based coding indicators adapted from Facione and Ennis for critical thinking and from Duval for visual representation. The results show that only 8 of 25 students (32%) solved all problems correctly and completely. Successful students demonstrated a structured thinking flow consisting of initial visualization, algebraic reasoning, and evaluation through visual verification. In contrast, less successful students tended to use formulas procedurally without connecting geometric meaning to algebraic forms. The findings indicate that, in conventional learning, critical thinking and geometric imagination are more likely to appear when students are explicitly accustomed to constructing sketches, translating geometric conditions into equations, and checking results visually. Practically, teachers need to scaffold initial sketching and visual verification; however, because this study is limited to one class and uses a qualitative descriptive design, the findings should be interpreted as a contextual mapping of thinking processes rather than a broad claim of instructional effectiveness.

Keywords: Analytic Geometry; Critical Thinking; Geometric Imagination; Circles and Tangent Lines; Visualization..

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INTRODUCTION

High school mathematics education is expected to develop Higher Order Thinking Skills (HOTS), including critical thinking and imaginative reasoning. Analytic geometry is a strategic domain for this purpose because students are required to coordinate algebraic expressions with spatial objects. In the topic of circles and tangent lines, students must interpret the position of a center, radius, axis, point, and tangent line; translate these conditions into equations; and then verify whether the symbolic result is geometrically meaningful. These requirements make analytic geometry an appropriate context for observing students' thinking processes rather than merely assessing their final answers.

Critical thinking in this study is understood as a purposeful and reflective process involving interpretation, analysis, inference, explanation, and evaluation (Facione, 1990), as well as the ability to give reasons, examine assumptions, and

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decide whether a mathematical conclusion is justified (Ennis, 2011). In analytic geometry, these components appear when students identify relevant information, connect geometric conditions with algebraic relations, choose a deductive strategy, and evaluate the consistency of the solution. Geometric imagination is interpreted through the lens of visual representation: students construct mental or written images of mathematical objects and coordinate them with symbolic representations. Duval (2006) emphasizes that mathematical understanding depends on the ability to convert and coordinate different representations; therefore, students' difficulty in analytic geometry may arise not only from algebraic errors but also from weak coordination between visual and symbolic registers.

Previous studies have frequently reported the benefits of visual and technology-based media in geometry learning. Syafari (2020) showed that an animation-based analytic geometry module supported students' critical thinking. Sugandi et al. (2022) reported that a GeoGebra-assisted analytic geometry module with the REACT strategy trained critical thinking through dynamic visualization. Nadzeri et al. (2024) found that augmented reality improved spatial visualization in geometry learning, while Suparman et al. (2024) concluded that GeoGebra-assisted geometry learning had a positive effect on spatial visualization. These studies suggest that visual support can help students integrate geometric and algebraic concepts.

However, the literature also shows that the role of technology and visualization should not be treated as automatically effective in every condition. The effectiveness of GeoGebra, augmented reality, and other visual technologies depends on learning design, students' prior knowledge, teacher guidance, task characteristics, and the extent to which students actively coordinate visual and symbolic representations. Moreover, many previous studies are intervention-oriented and outcome-based: they compare pretest and posttest scores or evaluate the effectiveness of a digital medium. Such approaches are valuable, but they often provide limited information about how students actually move from reading a

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problem, constructing a sketch, formalizing algebraic relations, and verifying the result during problem solving.

This methodological tendency creates a theoretical and empirical gap. If visual technology supports learning but its effects are not uniform, then it is important to understand the underlying cognitive mechanism through which students coordinate visual and algebraic representations. In conventional learning contexts, where advanced digital media may not be available, teachers still need evidence about what students do cognitively when solving non-routine analytic geometry problems. Therefore, the urgency of the present study is not merely to claim that analytic geometry develops HOTS, but to explain when and how critical thinking and geometric imagination appear or fail to appear in students' written work and interview explanations.

The novelty of this research lies in mapping students' thinking flow in a conventional analytic geometry class and in identifying success and failure points in the coordination of visual and algebraic representations. This study contributes a contextual analytical model, called the Student Thinking Flow Model, which describes the stages of initial visualization, algebraic reasoning, and evaluation through visual verification. The model is proposed as a qualitative analytical framework for understanding students' problem-solving processes, not as a general causal model of instructional effectiveness.

Based on this background, the research questions are: (1) How is the flow of high school students' thinking processes in solving analytic geometry problems on circles and tangent lines during conventional learning? (2) How does conventional learning facilitate or limit students' critical thinking and geometric imagination in connecting visual and algebraic representations?

METHOD

This study employed a qualitative descriptive design. The design was selected because the purpose of the research was to describe and interpret students' thinking processes when solving non-routine analytic geometry problems, not to

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test the effectiveness of an instructional treatment. The focus was on how students used visual and algebraic representations, how they justified their strategies, and how they verified their answers.

The participants were 25 Grade XI Natural Science (IPA) students at SMA Santa Maria Cirebon in the 2024/2025 academic year. The class was selected using criterion-based purposive sampling because the students had completed prerequisite topics relevant to the study, including linear equations, the distance from a point to a line, circle equations, gradients, and tangent-line conditions. Based on teacher documentation and classroom observation, the class represented heterogeneous mathematical performance: some students were fluent in algebraic manipulation and geometric interpretation, whereas others still experienced difficulties in connecting diagrams and equations. This heterogeneity was important for mapping different patterns of thinking in one classroom context.

The selected class was also appropriate because all students experienced the same conventional learning setting, the same teacher explanation, and the same set of non-routine post-test problems. Thus, the class provided an information-rich context for examining how students' thinking processes appeared after the topic of circles and tangent lines had been taught. The use of purposive sampling is consistent with qualitative research, in which participants are selected because they can provide rich information related to the research questions (Palinkas et al., 2015).

From the 25 students, six students were selected for semi-structured interviews through a second stage of purposive sampling. The selection considered variation in written performance and representation patterns. Four students were chosen from the group who completed all four problems correctly to trace the full thinking flow, while two students were chosen from those who experienced conceptual or procedural difficulties to identify points of breakdown in visual-algebraic coordination. The subjects were coded as S1-S6 to maintain confidentiality. This selection was intended not to represent the population

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statistically, but to compare information-rich cases that revealed different ways of thinking.

Table 1. Profile and Rationale for Selecting the Six Interview Subjects

Subject	Written performance pattern	Selection rationale	Dominant thinking characteristic
S1	Complete solution on all problems	Represents a complete visual-algebraic-verification flow	Consistently begins with sketching and then formulates equations
S2	Complete solution on all problems	Represents strong deductive strategy use	Uses formal conditions such as discriminant = 0 and checks results
S3	Complete solution on all problems	Represents flexible use of tangent-line strategies	Connects gradient, angle, and tangent conditions
S4	Complete solution on all problems	Represents successful but more procedural reasoning	Uses formulas accurately and supports them with visual checking
S5	Partial/incomplete solution	Represents difficulty in translating visual conditions into algebra	Can draw a sketch but has weak symbolic formalization
S6	Partial/incomplete solution	Represents procedural use of formulas without geometric meaning	Relies on memorized formulas and has difficulty explaining why they apply

The main instruments were four non-routine essay problems on circles and tangent lines, a semi-structured interview guide, and an observation sheet. The essay problems were designed to elicit the integration of geometric conditions and algebraic representations. They required students to identify known elements, construct or imagine a sketch, determine relevant geometric relations, formulate equations, and verify whether the answer was consistent with the problem context. The interview guide included questions such as: “How did you first understand the problem?”, “Why did you choose this strategy?”, “How did the sketch or mental image help you?”, and “How did you check your answer?”

Content validation was conducted through expert judgment before data collection. The validators consisted of three experts: one mathematics education lecturer, one analytic geometry lecturer/content expert, and one experienced senior high school mathematics teacher. The validation focused on six aspects: (1) alignment between problem content and research objectives, (2) appropriateness of

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the problems for Grade XI students, (3) clarity of mathematical language and instructions, (4) potential of the problems to elicit critical thinking, (5) potential of the problems to elicit geometric imagination and visual representation, and (6) suitability of the interview questions for tracing students' thinking processes. Revisions were made based on expert comments before the instruments were administered.

To avoid purely descriptive interpretation, the coding process used theory-based indicators. Critical thinking indicators were adapted from Facione's interpretation-analysis-inference-evaluation framework and Ennis' emphasis on reasoning, justification, and reflective judgment. Geometric imagination indicators were adapted from Duval's theory of coordination and conversion among representations. The indicators were operationalized as observable evidence in students' written work and interview responses, as shown in Table 2.

Table 2. Theory-based Operational Indicators Used in Data Analysis

Construct	Code	Operational indicator	Evidence in written work	Evidence in interview
Critical thinking	CT1	Identifying known information and the question precisely	Writes center, radius, tangent condition, line, or point correctly	Explains what was first understood from the problem
Critical thinking	CT2	Analyzing geometric-algebraic relations	Translates "tangent to x-axis" into radius relation or "tangent line" into discriminant/distance condition	Explains the reason for choosing the relation
Critical thinking	CT3	Selecting and applying a relevant deductive strategy	Uses substitution, distance formula, gradient, angle relation, or discriminant systematically	Explains why a strategy is more appropriate than guessing
Critical thinking	CT4	Evaluating and verifying the result	Checks whether the solution satisfies the problem and diagram	Explains how the result was checked
Critical thinking	CT5	Providing logical	Writes reasons, not only final formulas	Gives coherent verbal justification

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Construct	Code	Operational indicator	Evidence in written work	Evidence in interview
		explanation or justification		
Geometric imagination	GI1	Constructing an initial sketch or mental image	Draws circle, axis, center, radius, or tangent line	Says that drawing/imagination helped understand positions
Geometric imagination	GI2	Converting verbal information into visual and symbolic representations	Transforms text into sketch and equation	Explains movement from words to drawing/equation
Geometric imagination	GI3	Coordinating sketch and equation	Uses sketch to determine sign, position, distance, or feasibility	Explains the connection between diagram and formula
Geometric imagination	GI4	Using visualization for verification	Returns to sketch after calculation	States that the picture was used to check the answer

After the conventional instruction on circles and tangent lines was completed, all 25 students solved the four non-routine essay problems manually. Their written answers were collected as the primary data. Classroom observations were conducted to record students' use of sketches, diagrams, and visual reasoning during learning and problem solving. Semi-structured interviews were then conducted with the six selected subjects to clarify their written work, strategies, justifications, and use of visual imagination.

Data were analyzed using the interactive model of Miles et al. (2014), which includes data condensation, data display, conclusion drawing, and verification. First, students' written responses were reduced by identifying correct solutions, incomplete solutions, and common error patterns. Second, the responses were coded using the CT and GI indicators in Table 2. Third, interview transcripts were used to confirm or refine the interpretation of students' written work. Fourth, cross-case comparison was conducted to compare the six interview subjects, especially the differences between students who completed all problems and those who experienced difficulty.

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The trustworthiness of the analysis was strengthened through triangulation of written work, interview data, and classroom observation. Member checking was conducted by confirming key interpretations with the interviewed students. An audit trail was maintained through coding notes and analytic memos. Student identities were anonymized using subject codes, and all data collection procedures were carried out with school permission and student consent.

RESULT AND DISCUSSION

The post-test results showed that the overall success rate on the four analytic geometry problems was relatively low. Only 8 of 25 students (32%) solved all four problems correctly and completely, while 17 students experienced difficulty in one or more problems. The common difficulties included failing to formulate a circle equation when information was implicit, misunderstanding the condition for a line to be tangent to a circle, using a formula without recognizing its geometric meaning, and making algebraic manipulation errors. This result suggests that the topic did not merely demand procedural knowledge; it required students to coordinate visual representation, algebraic formalization, and reflective verification.

Successful students did not simply apply formulas. Their written work showed a sequence of indicators. First, they identified essential information in the problem, such as the center, axis, point, line, and tangent condition (CT1). Second, they translated geometric conditions into algebraic relations (CT2), for example, by interpreting "tangent to the x-axis" as the distance from the center to the x-axis being equal to the radius. Third, they selected appropriate deductive strategies (CT3), such as substitution of a point into a circle equation, using the distance from a point to a line, or applying the discriminant condition for tangency. Finally, they evaluated the plausibility of their answer by returning to the sketch or mental image (CT4 and GI4).

Given a circle centred at (4,-3) tangent to the x-axis (**Figure 1**). Successful students recognized that the centre lies below the x-axis at $y = -3$. They inferred that the distance from the center to the x-axis is $|-3| = 3$, which means the radius

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is $r = 3$. They then formed the circle equation with center $(4, -3)$ and radius 3: $(x-4)^2 + (y+3)^2 = 9$, to find the intersection with the x-axis, they substituted $y = 0 : (x-4)^2 + 9 = 9 \rightarrow (x-4)^2 = 0 \rightarrow x = 4$

The sketch in **Figure 1** illustrates the role of initial visualization. It is not only a drawing; it functions as evidence of GI1 because the student constructs a visual image of the circle, center, radius, and axis. It also supports CT2 because the student infers from the sketch that the radius is the distance from the center to the x-axis. Thus, the figure is directly linked to the operational indicators used in the analysis.

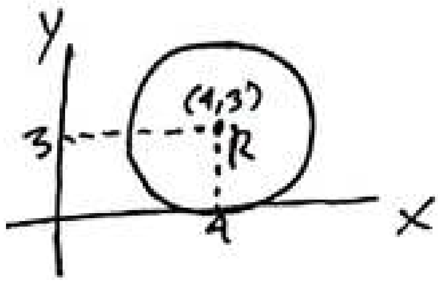

Figure 1. Initial Student Sketch Linked to GI1 (constructing a visual image) and CT2 (translating a tangent condition into a radius relation).

The sketch in **Figure 2** shows a more complex example. Students used the point-to-line distance formula for center (α, β) to the line $4x - 3y = 0$:

$$\frac{|4\alpha - 3\beta|}{\sqrt{16+9}} = \beta \rightarrow 5\beta = \pm(4\alpha - 3\beta)$$

From the two possible signs, students chose the solution consistent geometrically and obtained $\alpha = 2\beta$. Then, using the fact that the circle $(x-\alpha)^2 + (y-\beta)^2 = \beta^2$. The students passes through the point $(4, 5\frac{1}{3}) = \left(4, \frac{16}{3}\right)$:

$$(4-\alpha)^2 + \left(\frac{16}{3} - \beta\right)^2 = \beta^2$$

Substituting $\alpha = 2\beta$ and simplifying yields-:

$$36\beta^2 - 240\beta + 400 = 0 \rightarrow (6\beta - 20)^2 = 0 \rightarrow \beta = \frac{10}{3}$$

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Thus $\alpha = \frac{20}{3}$. Therefore, the circle equation is: $\left(x - \frac{20}{3}\right)^2 + \left(y - \frac{10}{3}\right)^2 = \frac{100}{9}$

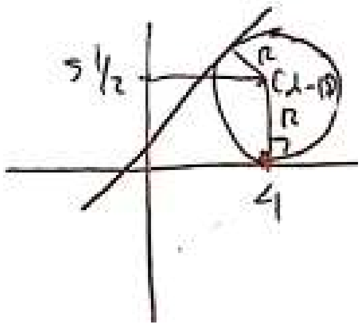

Figure 2. Student Sketch for a Tangent-Circle Problem Linked to GI1, GI3, and CT4

This figure indicates GI1 because a mental or written image of the configuration is constructed; GI3 because the sketch is coordinated with distance formulas and tangent conditions; and CT4 because the sketch is used to select the geometrically consistent solution among possible algebraic results.

Table 3. In-depth interpretation of critical thinking indicators in students' solutions

Indicator	Observed student action	Why the student may think this way	Interpretive meaning
CT1: Problem clarification	Students identify center, radius, line, point, and tangent condition before calculating.	They recognize that analytic geometry problems cannot be solved only by substituting numbers; the meaning of each condition must be clarified first.	Students begin to control the problem situation rather than react mechanically to formulas.
CT2: Geometric-algebraic relation	Students convert "tangent to x-axis" into a radius relation or "tangent line" into discriminant/distance condition.	The visual relation helps them see that tangency means one contact point or equal distance to the center.	Critical thinking appears as conceptual translation, not as formula memorization.
CT3: Deductive strategy	Students choose substitution, distance formula, gradient, or	They select a strategy because it logically follows	The solution becomes

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Indicator	Observed student action	Why the student may think this way	Interpretive meaning
	discriminant based on the problem condition.	from the geometric meaning of the condition.	systematic and justified.
CT4: Evaluation	Students compare the obtained equation or tangent line with the sketch.	They realize that an algebraically obtained answer may still be geometrically inconsistent if not checked visually.	Visual verification functions as a reflective control mechanism.
CT5: Explanation	Students explain why a discriminant is zero or why a radius equals a distance.	They move beyond procedural steps and attempt to justify the mathematical relation.	The written and oral explanations reveal the depth of critical thinking.

The interviews clarified that the main difference between successful and less successful students was not the presence of a formula, but the quality of coordination among representations. S1, S2, S3, and S4 were able to connect a sketch with an algebraic relation and then use the sketch again to check the answer. S5 could create a sketch but had difficulty translating it into a valid equation, while S6 tended to recall formulas without being able to explain the geometric meaning. This contrast shows that geometric imagination supports critical thinking only when the visual image is connected to algebraic reasoning and evaluation.

Table 4. Cross-case Comparison of The Six Interview Subjects Based on CT and GI Indicators

Subject	**Dominant CT indicators**	**Dominant GI indicators**	**Main strength**	**Main difficulty or limitation**
S1	CT1, CT2, CT3, CT4, CT5	GI1, GI2, GI3, GI4	Complete flow from sketch to equation and visual checking	Needs time to write complete justifications
S2	CT2, CT3, CT4	GI2, GI3, GI4	Strong use of discriminant/distance conditions for tangency	Sketch is sometimes less detailed
S3	CT1, CT2, CT3	GI1, GI2, GI3	Flexible in connecting angle,	Verification is present but not

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Subject	Dominant CT indicators	Dominant GI indicators	Main strength	Main difficulty or limitation
			gradient, and tangent-line conditions	always explicitly written
S4	CT2, CT3, CT4	GI1, GI3, GI4	Accurate procedural use supported by visual checking	Explanation tends to be formula-centered
S5	CT1, partial CT2	GI1, partial GI2	Able to begin with a sketch and identify given information	Difficulty converting sketch into correct algebraic relation
S6	Partial CT1, partial CT3	Limited GI1	Recognizes some formulas related to circle and tangent line	Uses formulas mechanically and cannot explain discriminant/distance meaning

This comparison indicates that the Student Thinking Flow Model was most visible in S1-S4. They began by forming a visual image, moved to algebraic formalization, and ended with evaluation. In contrast, S5 showed an interrupted flow between visualization and algebraic reasoning, while S6 showed an interrupted flow at the beginning because the visual representation was weak. Thus, the model should be understood as a pattern found in successful reasoning and as a diagnostic tool for locating where students' thinking breaks down.

The overall findings can be formulated into the Student Thinking Flow Model shown in Figure 3. The first stage, initial visualization, is marked by GI1, GI2, and CT1 because students construct a sketch or mental image and identify important information. The second stage, algebraic reasoning, is marked by CT2, CT3, and GI3 because students transform visual conditions into equations, gradients, distances, or discriminant relations. The third stage, evaluation and visual verification, is marked by CT4, CT5, and GI4 because students check whether the algebraic result fits the original geometric situation.

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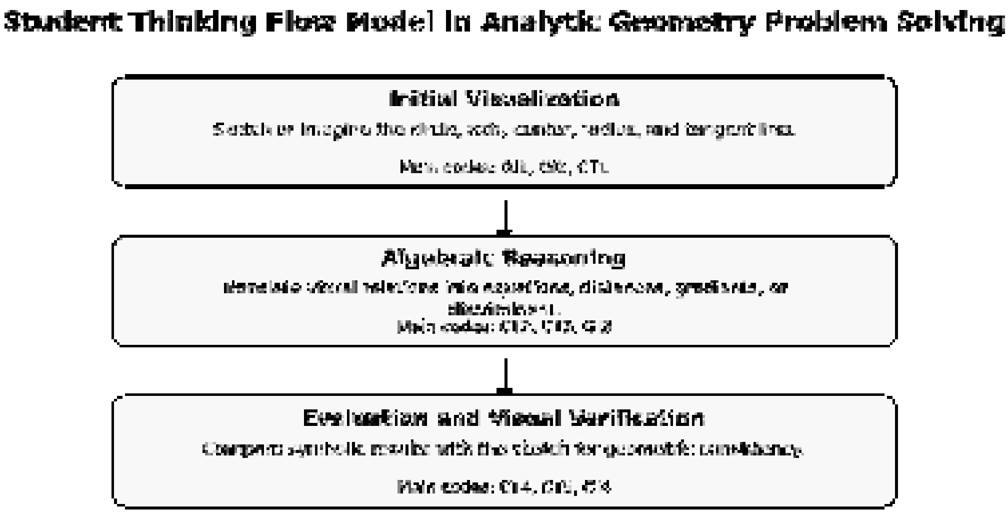

Figure 3. Revised Student Thinking Flow Model with Integrated Critical Thinking and Geometric Imagination Indicators.

The findings support earlier studies that highlight the importance of visualization in mathematics learning, such as Syafari (2020), Sugandi et al. (2022), Nadzeri et al. (2024), and Suparman et al. (2024). However, this study differs from those intervention-based studies because it does not claim that a particular medium improves learning outcomes. Instead, it shows how students build visual-algebraic connections in a conventional classroom setting. This difference is important because many studies on GeoGebra, AR, and visual technologies emphasize final achievement scores, while the present study traces the process through which the relationship between diagram and equation is constructed.

The results also refine the assumption that visualization automatically leads to better critical thinking. The cross-case analysis shows that drawing alone is insufficient. S5, for example, could begin with a sketch but still had difficulty formalizing the correct equation. This indicates that visualization must be accompanied by explicit conversion from visual representation to algebraic representation. In Duval's terms, mathematical understanding requires coordination

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among representations; therefore, students need guidance not only to draw but also to interpret what the drawing means algebraically.

The limited mastery rate of 32% suggests that conventional learning can facilitate the emergence of critical thinking and geometric imagination, but the effect is uneven. Students who received and internalized repeated practice in sketching and verification were more able to complete the problem-solving cycle. Students who relied on formulas without visual meaning tended to stop at procedural manipulation. This finding explains the scientific urgency of the study: teachers need to know where students' thinking processes succeed or break down so that instructional support can target the coordination of representations, not only the memorization of formulas.

Practically, the findings suggest that teachers should make sketching and visual verification explicit parts of analytic geometry problem solving. Before manipulating equations, students should be asked to represent the problem situation visually. After obtaining the result, students should compare the answer with the sketch to check whether the solution is geometrically reasonable. Such activities may support critical thinking because students must justify, infer, and evaluate; they may also support geometric imagination because students repeatedly coordinate mental images, diagrams, and symbolic expressions.

CONCLUSION

This qualitative descriptive study shows that, in the class studied, successful solutions to non-routine analytic geometry problems on circles and tangent lines tended to follow a structured thinking flow: initial visualization, algebraic reasoning, and evaluation through visual verification. This flow appeared clearly among students who solved all problems correctly and was less complete among students who experienced difficulties. The key distinction between successful and less successful students was the quality of coordination between visual and algebraic representations, not merely the ability to recall formulas.

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The main contribution of this study is the contextual mapping of students' thinking processes and the formulation of the Student Thinking Flow Model as an analytical framework for reading students' problem-solving work. The model contributes to the literature by showing how critical thinking and geometric imagination interact during conventional analytic geometry learning. It also identifies specific points of difficulty: weak translation from geometric conditions to algebraic relations, weak understanding of tangent conditions, and limited visual verification.

The conclusion should not be interpreted as a broad claim that conventional analytic geometry learning automatically develops HOTS for all students. Because the study involved one class, six interview subjects, and a qualitative descriptive design without pretest-posttest comparison, the findings are context-bound and analytically transferable rather than statistically generalizable. Future studies may test the Student Thinking Flow Model in broader contexts, compare conventional and technology-supported learning using quasi-experimental designs, and use think-aloud protocols or eye-tracking to examine students' visual-algebraic coordination in more detail.

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