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CONTROVERSIAL REASONING LEVELS AMONG PROSPECTIVE MATHEMATICS TEACHERS IN REAL ANALYSIS PROOF CONSTRUCTION: A QUALITATIVE CASE STUDY

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Abstrak

Konstruksi bukti dalam Analisis Real menuntut navigasi konflik kognitif yang kompleks, namun mekanisme yang menjembatani keyakinan intuitif dan logika formal masih belum banyak dieksplorasi. Penelitian ini mengkaji dinamika penalaran kontroversial di kalangan calon guru matematika dalam mengonstruksi bukti untuk limit barisan bernilai real. Dengan menggunakan pendekatan studi kasus kualitatif di sebuah universitas di Kediri, data dikumpulkan dari 40 mahasiswa sarjana melalui tugas masalah kontroversial dan wawancara semi-terstruktur. Analisis data mengikuti kerangka Miles dan Huberman, yang meliputi reduksi data, penyajian data, dan penarikan kesimpulan, dengan menelusuri trajektori kognitif pada tiga tingkatan: awal, eksplorasi, dan klarifikasi. Hasil penelitian mengungkapkan bahwa penalaran pada tingkatan awal dibatasi oleh miskonsepsi intuitif, di mana mahasiswa mengenali kontradiksi tetapi memberikan justifikasi yang tidak relevan secara konseptual. Tingkatan eksplorasi muncul sebagai tahap yang kritis namun terfragmentasi, di mana mahasiswa mulai menginisiasi strategi formal seperti induksi matematika, namun gagal mengintegrasikan prasyarat esensial bagi konvergensi, sehingga menghasilkan kesimpulan yang bias. Tingkatan klarifikasi dicirikan oleh kemampuan merekonstruksi konsistensi logis melalui penggunaan bukti kontradiksi yang tepat. Pola kognitif yang berbeda yang teramati pada ketiga tingkatan tersebut mengindikasikan bahwa penalaran kontroversial merupakan proses yang dinamis dan berkembang, bukan sebuah klasifikasi yang statis, sebagaimana tercermin dari perbedaan kualitatif dalam struktur argumentasi dan strategi bukti mahasiswa pada setiap tingkatan.

Kata kunci: Konflik Kognitif; Kontradiksi; Pembuktian Matematis; Penalaran Kontroversial

Abstract

Constructing proofs in Real Analysis requires navigating complex cognitive conflicts, yet the mechanisms that bridge intuitive beliefs and formal logic remain underexplored. This study investigates the dynamics of controversial reasoning among prospective mathematics teachers as they construct proofs of the limits of real-valued sequences. Employing a qualitative case study at a university in Kediri, data were collected from 40 undergraduate students through controversial problem tasks and semi-structured interviews. Data analysis followed the Miles and Huberman framework, comprising data reduction, data display, and conclusion drawing, tracing cognitive trajectories across three levels: initial, exploration, and clarification. Results reveal that reasoning at the initial level is constrained by intuitive misconceptions, in which students recognise contradictions but offer conceptually irrelevant justifications. The exploration level emerges as a critical yet fragmented stage in which students initiate formal strategies, such as mathematical induction, but fail to integrate essential prerequisites for convergence, leading to biased conclusions. The clarification level is characterised by the ability to reconstruct logical consistency through the appropriate use of proof by contradiction. The distinct cognitive patterns observed across the three levels indicate that controversial reasoning constitutes a dynamic, evolving process rather than a static classification, as evidenced by qualitative differences in students' argumentation structures and proof strategies at each level.

Keywords: Cognitive Conflict; Contradiction; Controversial Reasoning; Mathematical Proof

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INTRODUCTION

Proving in mathematics serves as a fundamental indicator of proficiency in deep conceptual knowledge, particularly for future mathematics teachers who must master the expression of formal notions and mathematical statements (Alcock & Weber, 2005; Isran et al., 2025; Moralı & Filiz, 2023; Perangin-angin et al., 2024; Wasserman & Weber, 2017). While this rigorous process demands command of definitions, theorems, and logical forms (Hamami & Morris, 2024; Stefanowicz, 2014), as well as critical deductive thinking (Moralı & Filiz, 2023; Sommerhoff et al., 2021), the transition to such formal axiomatic systems often precipitates cognitive conflict. This conflict inevitably arises when established mathematical truths grounded in agreed definitions (Andreoli, 2001; Durand-Guerrier, 2008; Hals, 2020; Hamami & Morris, 2024; Sterpetti, 2020) challenge students' preconceived beliefs. The resulting tension between prior knowledge and contradictory new information (Hr et al., 2023; Lee & Chen, 2008; Swan et al., 2006; Tall, 1977; Zaslavsky et al., 2012) manifests as a cognitive clash that disrupts initial understanding (Rolka et al., 2007; Watson, 2002, 2007). Consequently, resolving this dissonance requires deep reflection and conceptual modification, ultimately leading to a shift in perspective or the clarification of existing concepts (Adnyani, 2020; Lombardi et al., 2016).

In the academic context, particularly within Real Analysis, constructing valid proofs presents a significant hurdle due to the subject's abstract and complex nature (Arnellis et al., 2023; Chand, 2021; Delgado-Rebolledo & Zakaryan, 2020; Geisler & Rolka, 2021; Isran et al., 2025; Welleck et al., 2021; Wijaya et al., 2023). These difficulties are fundamentally rooted in weak mastery of foundational concepts, limited mathematical reasoning skills, and a distinct lack of experience with tasks that require reflective thinking (Chand, 2021; Herizal, 2020; Minggi et al., 2017; Rambe et al., 2024; Siregar, 2016; N. F. Siregar, 2020). To mitigate these deficits, cognitive conflict can be deliberately elicited in educational settings through "controversial problems"—ambiguous tasks that mimic everyday debates involving contrasting viewpoints (Mueller & Yankelewitz, 2014; Simic-Muller et al., 2015). By presenting these contradictions, educators challenge students to think more critically and reflectively, transforming passive confusion into active inquiry (Subanji et al., 2021, 2023).

However, the emergence of cognitive conflict is a double-edged sword; without proper management, it can block progress rather than encourage deeper engagement in creating proofs (El Walida & Sa’dijah, 2022; Hr et al., 2023; Ngicho et al., 2020; Rosyadi et al., 2024; Subanji et al., 2021; Sutopo, 2021; Swan et al., 2006). To handle these situations effectively, students must develop "controversial reasoning," which involves identifying contradictions, exploring their causes, and building logical arguments to resolve them (El Walida & Sa’dijah, 2022; Muniri & Musrikah, 2024; Rosyadi, Sa’dijah, et al., 2021; Subanji et al., 2021). Subanji et al. (2021) formalised this process into three levels: the initial level (noticing contradiction without explanation), the exploration level (investigating causes and alternatives), and the clarification level (constructing valid consistency). However, empirical evidence suggests that most students—especially in Real Analysis—stagnate at the exploration level, with very few reaching the clarification stage (El Walida et al., 2024; Rosyadi, Sa'dijah, et al., 2021; Rosyadi et al., 2024; Rambe et al., 2024). Ultimately, fostering this reasoning capacity is essential, as it allows students not only to recognise contradictions but to construct the clarifying arguments necessary for valid proof.

Prior research has firmly established the pedagogical value of controversial problems in mathematics education. Existing literature highlights their effectiveness in promoting Higher-Order Thinking Skills (HOTS) and creative thinking processes (Rosyadi, Sadijah, et al., 2021; Suryawan & Ratnaya, 2023; Subanji et al., 2023). Furthermore, scholars have expanded this domain by examining the psychological dimensions, such as prospective teachers' perceptions (Susiswo et al., 2022) and the interplay between controversial problems and metacognitive strategies (El Walida & Sa'dijah, 2022; Swastika et al., 2022). More recently, studies have begun to integrate these problems into specific contexts like ethnomathematics (Suryawan et al., 2023), and cognitive dynamic analysis (Dewi et al., 2024). Collectively, these studies provide a robust foundation for the benefits of controversial problems. However, they predominantly focus on general thinking skills rather than on the rigorous mechanisms of formal mathematical validation.

Despite this growing body of literature, a critical gap remains regarding the application of controversial reasoning in advanced mathematical proof. Most current research is confined to secondary school levels or focuses on algebraic contexts and general conceptual understanding among undergraduates (Susiswo et al., 2023; Muniri & Musrikah, 2024; El Walida et al., 2024; Rosyadi et al., 2024). However, research explicitly investigating controversial reasoning within the high-stakes context of Real Analysis proof construction remains scarce. This absence is significant because Real Analysis—particularly the concept of limits—is notoriously counter-intuitive and prone to epistemological obstacles (Arnellis et al., 2023; Isran et al., 2025). Students frequently face internal contradictions between their concept images and formal definitions but lack the reasoning tools to resolve them (Chamberlain & Vidakovic, 2021; Hamdani et al., 2023; Quarfoot & Rabin, 2022; Stylianides, 2007). The existing literature has identified that students struggle. However, it has not yet been sufficiently unpacked how the trajectory of controversial reasoning serves as a mechanism for bridging intuition and formal proof.

To address this gap, this study investigates controversial reasoning specifically within the context of the limits of real number sequences. While grounded in the theoretical framework of reasoning levels (Initial, Exploration, Clarification) proposed by Subanji et al. (2021), the novelty of this study lies in deepening the context by shifting the focus from general problem-solving to formal proof. Unlike prior studies that tend to focus on reasoning skills in problem-solving contexts, this study seeks to uncover the thinking processes and specific obstacles students encounter at each level of mathematical proof. By tracing how students bridge their intuition of limits and formal definitions, this research provides a detailed trajectory of the proof-construction process. These findings are expected to offer new theoretical insights and serve as a reference for educators in scaffolding students' transition from intuitive to formal understanding.

METHOD

This study aims to describe and explore the characteristics of the levels of controversial reasoning exhibited by undergraduate students in the Mathematics Education (*Tadris Matematika*) program. The levels include initial, exploration, and clarification. The focus is on constructing proofs related to the limits of real sequences in real analysis. Accordingly, a qualitative approach was adopted. This approach was selected because it allows for in-depth investigation of complex phenomena (Patton, 2014), such as controversial reasoning, which cannot be adequately captured through quantitative measures. The primary objective is to develop a comprehensive understanding of how students construct and navigate contradictions in mathematical proof—an inherently cognitive activity that requires interpretive and contextual analysis. The analytical framework of controversial reasoning levels developed by Subanji et al. (2021) (see Table 1) was employed as a lens to categorise and interpret the distinctive patterns of student thinking when confronted with ambiguous or cognitively conflicting mathematical situations.

Table 1. Levels of Controversial Reasoning in Mathematical Problem Solving

Controversial Reasoning Level	Characteristics
Initial	• Recognises the existence of a contradiction between encountered facts and prior knowledge • Unable to identify the component causing the contradiction
Exploration	• Acknowledges the presence of a contradiction • Identifies components contributing to the contradiction • Unable to proceed with reasoning that leads to a valid solution
Clarification	• Clarifies the components and sources of the controversy • Develops a mathematically and logically sound resolution • Constructs justifiable arguments to validate the resolution • Appropriately applies known concepts to reach a valid conclusion

This study involved 40 fourth-semester students from the Mathematics Education Program at UIN Syekh Wasil Kediri during the 2023 academic year. All participants had completed coursework in algebraic principles, calculus limits, and elementary number-theory proofs. Participants were selected using purposive sampling with a criterion-based approach (Palys, 2008), ensuring the inclusion of representatives from each level of controversial reasoning.

Based on written test results, students were classified into three reasoning levels: intial ($n = 15$), exploration ($n = 24$), and clarification ($n = 1$). From each level, one subject was selected for an in-depth interview using two operationally defined criteria. The first criterion was response consistency, defined as the degree to which a student's written argumentation aligned coherently with their verbal explanations during a preliminary screening interview, verified through direct comparison of written codes and interview transcripts. The second criterion was indicator representativeness, defined as the extent to which a student's response exhibited all defining characteristics of their assigned reasoning level, as specified in Table 1, without displaying partial or transitional features of an adjacent level. Students whose responses showed mixed indicators across two levels were excluded from selection. The decision to select one subject per level is consistent with the exploratory aims of this study: rather than confirming patterns across multiple cases, the analysis seeks to construct a detailed cognitive portrait of each reasoning level, following the logic of *typical case purposive sampling* (Patton, 2014) in which a single well-chosen case can illuminate the defining features of a category. The inherent limitation of this single-case-per-level design — namely, the reduced ability to confirm intra-level variation — is acknowledged.

The qualitative data collected comprised students' written responses and verbatim interview transcripts. The research instrument consisted of a controversial mathematical task explicitly designed to trigger cognitive conflict by presenting a proof argument that appeared logically sound but led to a contradictory or false conclusion (Subanji et al., 2021). The task required students to engage with a sequence-limit proof whose reasoning steps appeared intuitively correct. However, it was formally flawed, as illustrated in Figure 1, thereby enabling the researchers to observe how students identify, navigate, and resolve logical inconsistencies in the proof-construction process.

To establish the instrument's content validity, the task underwent expert review prior to data collection. Two lecturers in mathematics education, each with expertise in Real Analysis and mathematics proof instruction, independently evaluated the instrument. Each expert assessed the relevance and appropriateness of the task against two dimensions: (1) its capacity to elicit genuine cognitive conflict in the target construct of controversial reasoning, and (2) its alignment with the theoretical indicators of the three reasoning levels (initial, exploration, clarification) as defined by Subanji et al. (2021). Expert judgments were recorded on a four-point rating scale (1 = not relevant, 4 = highly relevant), and content validity was computed using the Gregory formula, yielding a Content Validity Index (CVI) of 0.82. This value exceeds the minimum threshold of 0.75

recommended for acceptable content validity (Gregory, 2000), confirming that the instrument adequately measures the intended construct.

Following the completion of the task, each selected participant engaged in a semi-structured interview, during which they were asked to elaborate on their thought processes, explain their reasoning, and reflect on the elements they found controversial or challenging. All interviews were audio-recorded and transcribed verbatim for analysis.

A student is facing a problem involving a sequence of real numbers. **He is** given a recursive sequence for all $n \in \mathbb{N}$ with $x_1 = \frac{1}{2}$ and $x_{n+1} = x_n + \frac{n-1}{n+1}$.
He is asked to investigate the convergence of the given recursive sequence. Using **Theorem 3.1.9 – The Sequence Limit Theorem**, he believes that $\lim(x_n) = \lim(x_{n+1})$. Therefore, he assumes the value of $\lim(x_n)$ exists, namely x.
However, he becomes confused when the result he obtains is as follows:

$$\lim(x_{n+1}) = \lim\left(x_n + \frac{n-1}{n+1}\right)$$

$$\lim(x_{n+1}) = \lim(x_n) + \lim\left(\frac{n-1}{n+1}\right)$$

$$x = x + 1$$

$$0 = 1$$

a. In your opinion, does the student's answer make sense? Explain your opinion logically and in detail.
b. If you were acting as a peer tutor for the student, what could you explain regarding the problem so that he can understand it well?

Figure 1. Explicit Controversial Problem

Data analysis followed the Miles & Huberman (1994) framework, comprising three iterative and interactive stages: data reduction, data display, and conclusion drawing/verification. During data reduction, students' written responses and interview transcripts were systematically selected, simplified, and coded by the first researcher, with codes organised around three analytical foci: the level of controversial reasoning, proof strategies employed, and argumentation patterns. The resulting codes were grouped into thematic clusters corresponding to the indicators of each reasoning level. Reduced data were then organised into comparison matrices and descriptive narratives, with verbatim excerpts to ensure analytical transparency.

To strengthen the trustworthiness of the coding process, two credibility measures were applied. First, *peer debriefing* (Lincoln & Guba, 1985) was conducted by a colleague in mathematics education, who independently reviewed the coding scheme, scrutinised the assignment of student responses to reasoning levels, and raised critical questions about interpretive decisions. The peer reviewer confirmed the coding scheme's consistency with the theoretical indicators. Second, *triangulation* between written test responses and interview transcripts was employed to cross-verify each student's reasoning level classification, ensuring that level assignments were not based solely on written evidence but were corroborated by verbal explanations. Conclusions were drawn by identifying consistent patterns across both data sources, and the overall credibility, dependability, and confirmability of the findings were established through thick description, in accordance with the trustworthiness criteria proposed by Lincoln and Guba (1985).

RESULTS AND DISCUSSION

This section presents the results of the controversial reasoning test and interview data, which were coded based on the indicators of controversial reasoning levels as proposed by (Subanji et al., 2021). Based on the analysis of test results from 40 fourth-semester students of the Mathematics Education Program at IAIN Kediri, the distribution of reasoning levels is shown in Figure 2. Three students were selected for in-depth interviews, each representing one of the three levels of controversial reasoning: initial ($n_1 = 15$ students), exploration ($n_2 = 24$ students), and clarification ($n_3 = 1$ students). The subject selection process also considered the consistency of responses and the similarity of indicators within the same reasoning level. Based on the coding results, seven aspects of reasoning were identified in students' responses to controversial problems in mathematical proof. These findings are presented in Table 2, which compares the seven aspects across the three levels of controversial reasoning and describes how each aspect manifests at each level.

Although the three-level classification proposed by Subanji et al. (2021) proved sufficient to accommodate all observed student responses — with no cases falling outside the defined levels — the present findings reveal a richer cognitive landscape within each level than the original framework describes. As elaborated in Table 2, the analysis identified six reasoning dimensions that emerged specifically from the context of proof construction in Real Analysis: awareness of contradiction, identification of the source of the conflict, exploration and proof strategies, use of formal mathematical concepts, ability to construct logical arguments, and reflection on the problem-solving process. These dimensions were not explicitly specified in Subanji et al.'s original model, which was developed from student responses to non-proof mathematical problems. Their emergence in the present study suggests that while the three-level structure remains structurally valid, its behavioural content is substantially shaped by the epistemic demands of the proof context in which it is applied.

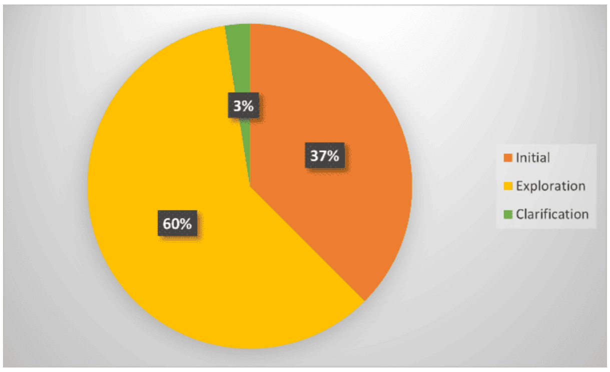

Figure 2. Distribution of Students Across Levels of Controversial Reasoning

Table 2. Cognitive Behaviours in Controversial Reasoning Levels

Reasoning aspect	Initial	Exploration	Clarification
Awareness of contradiction	Recognises that the result is unreasonable, but fails to identify the cause correctly.	Acknowledges the contradiction and begins to explain the cause more accurately.	Identifies the contradiction and connects it to a faulty logical assumption.
Identification of the conflict source	Provides incorrect and conceptually irrelevant reasons.	Mentions more logical causes, though not entirely consistent.	Accurately explains the source of the conflict.
Exploration and proof strategies	No further exploration; only vague planning without implementation.	Makes observations and uses induction but shows misconceptions in applying concepts.	Applies numerical strategies (with computational tools), conducts observations, and constructs a

Reasoning aspect	Initial	Exploration	Clarification
			complete and valid proof by contradiction.
Use of formal mathematical concepts	Misinterprets the concept of limits and sequence structure.	Applies induction and limit theorems, but with errors in inequalities and conclusions.	Correctly uses formal definitions, induction, and convergence theorems.
Ability to construct logical arguments	Fails to build a structured argument.	Begins forming logical arguments, but is incomplete due to misconceptions.	Constructs coherent, valid arguments using formal proof techniques.
Reflection on the problem-solving process	Suggests improvements, but fails to integrate them into the solution.	Reflects and attempts corrections, but exhibits bias in the conclusion.	Able to revise and refine the solution based on logical and mathematical reflection.
Resolution	It does not reach a correct solution or a valid proof.	The solution is incomplete, although the initial reasoning appears promising.	Successfully solves the problem in a logical, valid, and mathematically sound manner.

Based on SI's written response shown in Figure 3, the subject stated that the given answer was unreasonable and identified three sources of contradiction. However, none of the three was accurate or reflected the actual cause of the contradiction. First, SI claimed there was an error in using the limit property—specifically, the separation of limits in an addition expression. SI stated that this property was misused, yet the concept of limit theorems did not contradict the reasoning shown in the task. In other words, the step in question was mathematically correct and not the source of the contradiction. Second, SI stated that the limit of the sequence (x_{n+1}) was 0. This conclusion stemmed from a misinterpretation of the recursive form of the sequence (x_n) SI suggested dividing by $n + 1$, which was irrelevant to the intended goal of the task. Third, the subject claimed that the expression $\lim(x_n) = \lim\,(x_{n+1})$ was invalid, arguing that the equality only held for $n = 1$, and not for $n > 1$. However, the explanation provided was not conceptually relevant and could not be considered a valid justification for the contradiction.

SI: *Yes, I think the answer is strange because something seems inconsistent. The result does not seem logical. So I think there must be a mistake in the process.*

Q: *Okay. You listed three points you believe are the cause of the contradiction. The first is that the question misapplies the limit property, specifically the separation of limits. Can you explain that further?*

SI: *Yes, in that step, there is something like* $lim\left(x_n + \frac{n-1}{n+1}\right)$, *and it has directly split into* $lim(x_n) + lim\left(\frac{n-1}{n+1}\right)$ *I thought that was not allowed because I assumed that property could not be used in this expression.*

Q: *Why do you think that property cannot be used?*

SI: *Because I saw that sequence* (x_n) *is recursive, and the form of the sum looks different, so I thought it could not just be separated like that. I assumed that was where the contradiction came from.*

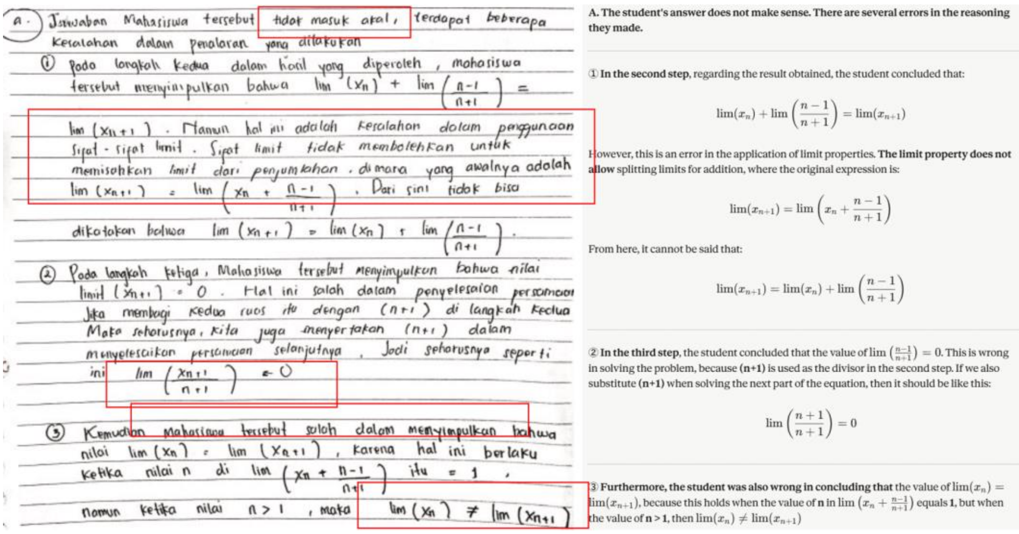

Figure 3. SI's Work

Furthermore, SI wrote down several intended plans of action, yet no attempts were made to justify or prove the argument. The listed plans included: demonstrating correct use of the limit property, explaining proper and consistent steps, identifying the appropriate method for determining the limit of a recursive sequence, and using Excel to verify whether $\lim(x_{n+1}) = \lim(x_n)$. These findings were confirmed through interview data and are coded and categorised in Table 3 and Table 4.

Table 3. Coded Reasoning Steps of the Subject at the Initial Level

No	Data Excerpt	Code
1	SI stated that the answer was unreasonable	Awareness of an implausible result
2	Identified three possible errors causing the contradiction, but all were incorrect.	Irrelevant identification of the source of conflict
3	Claimed an error in applying the limit property, though the concept was not violated.	Misconception about the limit property
4	Claimed that the limit of x_{n+1} is 0, due to misinterpreting the sequence form.	Misinterpretation of recursive sequence structure
5	Suggested dividing by $n+1$, which is contextually irrelevant.	Inappropriate remedial strategy
6	Stated that $\lim\,(x_n) = \lim\,(x_{n+1})$ was incorrect because it differs when $n=1$ and $n>1$.	Invalid reasoning for contradiction
7	Justifications were not mathematically valid.	Mathematically invalid arguments
8	Mentioned plans to provide examples and re-explain, but no concrete actions were taken.	Unexecuted plans for revision
9	Plans were general and disconnected from the problem context	Context-detached planning
10	Did not demonstrate any clarification or justification process in the written work.	Failure to construct logical arguments

Table 4. Reasoning Categories Identified in the Subject at the Initial Level

Emergent Reasoning Category	Related Codes
Initial awareness of contradiction	1
Irrelevant identification and explanation	2, 3, 4, 5, 6, 7
Reflective efforts not embedded in proof construction	8, 9
Inability to construct a consistent argument or clarification	10

Based on SE's response (see Figure 4), it was found that the subject acknowledged the conclusion was illogical. SE identified several flaws in the reasoning presented in the task and explained why the results were inconsistent with mathematical logic. First, SE stated that $\lim(x_n) = \lim\,(x_{n+1})$ is acceptable only when there is a recursive relationship. Second, SE argued that the assumption $\lim(x_n) = x$ is invalid without first proving convergence, implying that the existence of the sequence's limit cannot be assumed prematurely. These statements suggest that SE recognised a contradiction and accurately identified its underlying causes.

SE: *Yes, I noticed that the conclusion leads to a contradiction—for example, ending with* $0 = 1$, *which defies logic. So something must have gone wrong in the previous steps.*
Q: *Okay. You wrote that* $lim(x_n) = lim\ (x_{n+1})$ *can only be equated if the sequence is recursive. Can you explain this further?*
SE:*Because* x_{n+1} *directly depends on* x_n*If the sequence is convergent, the limit should generally be the same. However, this only applies if we already know the sequence is convergent.*
Q: *So you are aware that equating limits requires some conditions?*
SE: *Yes. If we do not know whether the limit exists, then we cannot just assume that* $lim(x_n) = x$.

In the next stage, SE attempted a formal proof by investigating whether the sequence converges. SE began by observing that the sequence is monotonically increasing. This was followed by a proof using mathematical induction. The base case $P(1)$ was successfully established. In the inductive step, assuming $x_k \leq x_{k+1}$, SE proved $x_{k+1} \leq x_{k+2}$ successfully concluding that the sequence is increasing. SE then attempted to prove the sequence is bounded above by $\frac{1}{2}$, using mathematical induction. While the base case and induction structure were addressed, SE made a conceptual error in dealing with inequalities. $\frac{n-1}{n+1} \leq \frac{1-1}{1+1}$ led to a flawed conclusion that $x_n \leq \frac{1}{2}$, and thus, that the sequence is bounded.

SE: *Yes. I used the recursive definition and noticed that* $x_{n+1} \geq x_n$ *for all* n*. So I concluded the sequence is increasing.*
Q: *Then you tried to prove it is bounded above by* $\frac{1}{2}$*. Could you walk me through your reasoning?*
SE: *I used the definition of* x_{n+1}*, then substituted* x_n *with its upper bound,* $\frac{1}{2}$*, which gives* $x_{n+1} \leq \frac{1}{2} + \frac{n-1}{n+1}$*. I simplified this and found it was still less than or equal to* $\frac{1}{2}$*, so I concluded it was bounded* $\frac{1}{2}$.
Q: *But you also used the value* $\frac{1-1}{1+1}$ *as the upper bound of* $\frac{n-1}{n+1}$*, and then concluded* $x_{n+1} \leq \frac{1}{2}$*. Are you sure that is valid?*
SE: *At the time, I assumed it was okay because the expression tends to* $\frac{1}{2}$.

This indicates that SE failed to recognise that the sequence was unbounded, resulting in a biased conclusion. SE then applied the Tail Limit Theorem and reached the contradictory result $0 = 1$, concluding that the sequence does not converge. However, SE failed to realise that an increasing and bounded sequence must be convergent. As a result, the contradiction was not recognised, and the reasoning process did not yield a valid solution. The findings from SE's work were coded and categorised as presented in Table 5 and Table 6.

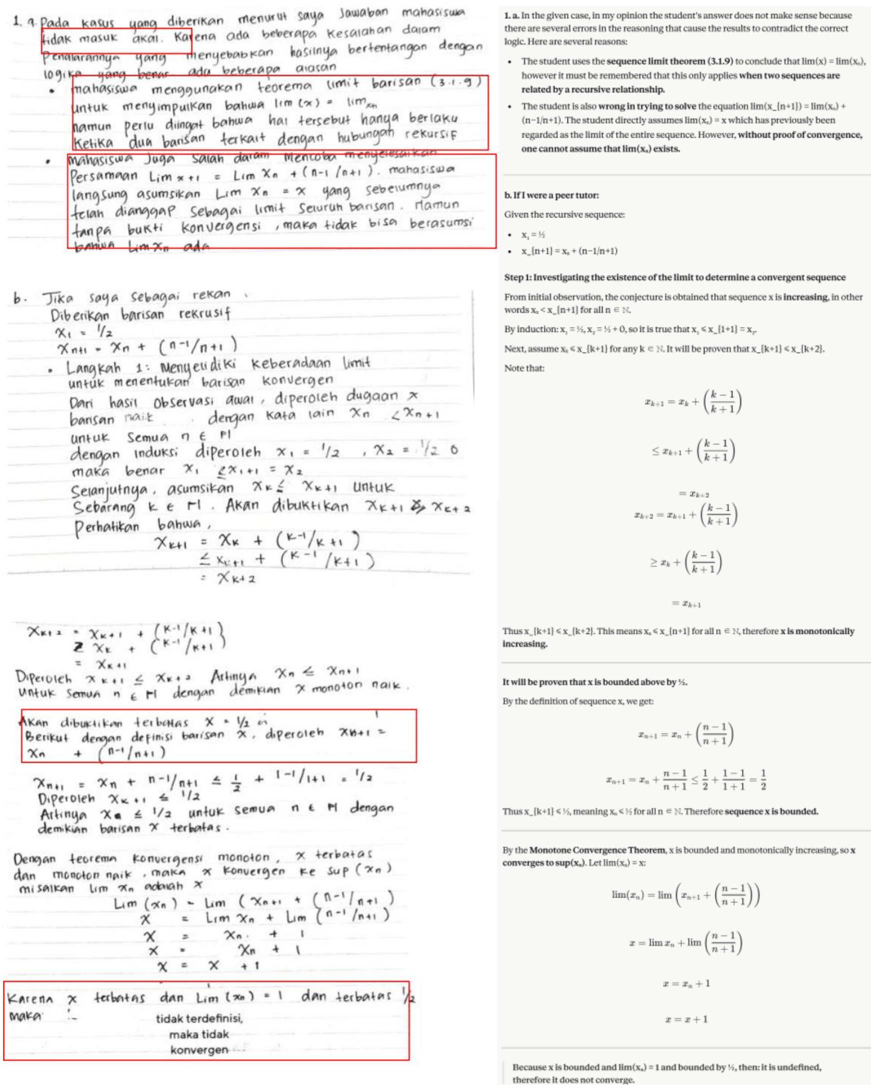

Figure 4. SE's work

Table 5. Coded Reasoning Steps of the Subject at the Exploration Level

No	Data Excerpt	Code
1	SE stated that the conclusion was illogical.	Recognition of an implausible result
2	$\lim(x_n) = \lim(x_{n+1})$ only if there is a recursive relationship	Partial understanding of limit properties
3	The assumption $\lim(x_n) = x$ is invalid without proof of convergence	Critique of assumptions
4	Accurately identified the source of the contradiction	Correct identification of contradiction
5	Investigated the existence of the limit	Exploration toward formal proof
6	Initial observation that the sequence is increasing	Numerical exploration strategy
7	Used mathematical induction for proof	Attempted logical reasoning
8	Successfully proved the sequence is increasing	Partial reasoning success
9	Conceptual error in inequality during induction	Logical flaw in proof technique
10	Biased conclusion: sequence is bounded above by $\frac{1}{2}$	Invalid conclusion
11	Applied the tail limit theorem	The theorem is applied without attention to prerequisites
12	Resulted in $0 = 1$ contradiction without noticing it	Failure to recognise contradiction
13	Did not realise that an increasing and bounded sequence must converge	Inability to connect the convergence theorem
14	Could not reach a valid solution	Incomplete reasoning process

Table 6. Reasoning Categories Identified in the Subject at Exploration Level

Emergent Reasoning Category	Related Codes
Initial awareness of contradiction	1, 3, 4
Early exploration and investigation	5, 6, 7
Partial success in reasoning	2, 8
Conceptual errors	9, 10, 13
Failure to draw valid conclusions	11, 12, 14

Based on SK's response (see Figure 5), it was found that the subject successfully recognised the contradiction by stating that the student's conclusion in the problem was illogical and that a contradiction appeared in the step leading to $0 = 1$. SK also noted that there were inappropriate steps taken and proposed that the sequence might not be convergent, thus rendering the assumption that the limit of (x_n) exists invalid. In this case, the subject could clarify both the components of the controversy and its source.

SK: *Yes, because the final step leads to the statement* $0 = 1$, *which is a contradiction and logically impossible. So, something must have gone wrong in the earlier steps.*
Q: *What do you think caused the contradiction?*
SK: *Most likely, the student assumed that* $lim(x_n) = x$, *even though there is no proof that the sequence* (x_n) *is convergent. So, I believe that assumption was incorrect.*

SK solved the problem by first conducting an initial exploration in MS Excel and recording the values in a table. He emphasised the importance of initial observation in identifying the sequence pattern. Based on the data, SK conjectured that the sequence is monotonically increasing and divergent (unbounded). Consequently, he argued that assuming $\lim(x_n) = x$ was inappropriate, and that this assumption led to the contradiction $0 = 1$. This indicates that SK could construct a coherent argument to justify the contradiction.

SK further argued that to solve the problem, one must prove that the sequence is unbounded. He began with a hypothesis and set out to prove that the sequence is unbounded. Using proof by contradiction, SK assumed the sequence was bounded, which implies the existence of a constant M such that $|x_n| = x_n < M$. From this, it follows that:

$$x_{n+1} - M > x_{n+1} - x_n = \frac{n-1}{n+1} > 0.$$

This implies $x_{n+1} > M$, contradicting the assumption that M is an upper bound. Therefore, the initial assumption is rejected, meaning the sequence is unbounded.

Q: *After that, you conducted an observation of the sequence. Can you explain what you did and your purpose?*
SK: *I calculated several initial terms of the sequence using MS Excel and organised the results into a table. The purpose was to identify the general form and make a conjecture. The data showed the sequence kept increasing and size, so I hypothesised it is unbounded.*
Q: *Based on this observation, did you conclude immediately that the sequence is divergent?*
SK: *Not yet. I understand that conjectures are insufficient, so I proceeded to prove that the sequence is unbounded. Because if it is unbounded, then it cannot be convergent.*
Q: *How did you construct the proof?*
SK: *I used the method of contradiction. I began by assuming the sequence is bounded, meaning there exists, M such that* $x_n < M$ *for all n. Then I showed that* $x_{n+1} - M > 0$, *it means* $x_{n+1} > M$, *which contradicts the assumption. So, I concluded that the assumption must be false, and therefore the sequence is unbounded.*
Q: *Do you think this argument is sufficient to conclude the sequence is divergent?*

***SK**: Yes. Because if a sequence is unbounded, it cannot be convergent. This also explains the contradiction, like* $0 = 1$*Because the original assumption was convergence, which is false.*

SK provided sound reasoning in his proof. Recognising that the sequence may not converge, he examined its boundedness. As a result, he could construct a mathematically sound and logical solution. SK correctly applied the definition of a bounded sequence and used the Monotone Convergence Theorem to show that the sequence does not converge because it is unbounded. This shows that SK could effectively clarify and apply relevant mathematical concepts, resulting in a valid resolution. The findings were coded and categorised in Table 7 and Table 8.

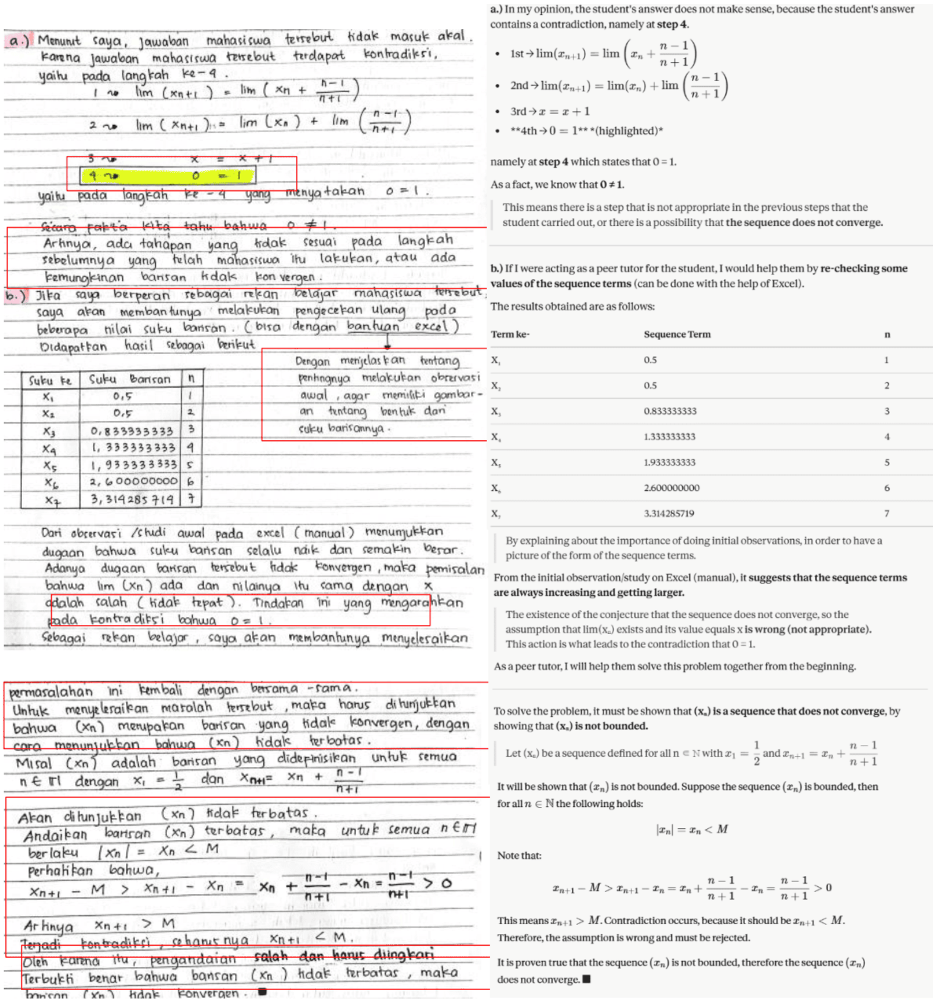

Figure 5. SK's work

Table 7. Coded Reasoning Steps of the Subject at the Clarification Level

No	Data Excerpt	Code
1	SK identified the contradiction, stating that the conclusion was illogical.	Contradiction identification
2	Noted that some steps were invalid, and the sequence may not converge	Recognition of an implausible result
3	Clarified the components and the source of controversy	Clarification of contradiction
4	Used Excel to observe the sequence	Numerical observation strategy
5	Conjectured that the sequence is increasing and unbounded	Conjecture based on empirical data
6	Rejected the assumption $\lim (x_n) = x$	Critique of the base assumption
7	Outlined proof via contradiction	Logical proof structure
8	Assumed boundedness, then showed a contradiction	Proof by contradiction method
9	Concluded that the assumption must be rejected	Logical conclusion
10	Correctly used the definition of a bounded sequence	Application of formal definition
11	Applied the Monotone Convergence Theorem	Theorem application
12	Produced a mathematically valid and logical resolution	Valid mathematical argumentation

Table 8. Reasoning Categories Identified in the Subject at Clarification Level

Emergent Reasoning Category	Related Codes
Identification and Clarification of Contradiction	1, 2, 3, 6
Strategy for Exploration and Observation	4, 5
Construction of Arguments and Proofs	7, 8, 9

A central finding of this study is not merely the confirmation of three controversial reasoning levels, but the elaboration of their cognitive content within a proof-based context. The original Subanji et al. (2021) model defines each level primarily in terms of contradiction awareness and resolution capacity. The present findings extend this by specifying the proof strategies, the application of formal concepts, and the reflective behaviours that characterise each level when students engage with an Explicit Controversial Problem in Real Analysis. This elaboration constitutes a theoretical contribution distinct from simple replication; it demonstrates that controversial reasoning is not a decontextualised cognitive phenomenon, but one whose behavioural expression is mediated by the mathematical domain and task type. Future studies might examine whether similar dimensional elaborations emerge in other proof-based domains, such as abstract algebra or topology, or whether the dimensions identified here are specific to the structure of Real Analysis proof tasks.

The differences in reasoning observed across levels are most clearly evidenced by how SI, SE, and SK each engaged with the same problem. SI detected an implausible result but consistently misattributed the contradiction to mathematically correct steps — including a valid application of the limit separation property — and made no actual attempt at a proof, producing only unexecuted plans. SE correctly diagnosed the convergence assumption as the root of the problem, initiated a structured proof via mathematical induction, yet produced a biased conclusion due to an inequality error in the inductive step. SK, by contrast, grounded conjecture in numerical observation through MS Excel before constructing a complete proof by contradiction, correctly applying the definition of bounded sequences and the Monotone Convergence Theorem. These contrasts reflect controversial reasoning involving cognitive conflict that triggers deep reflection to uncover the controversy's core (El Walida et al., 2024; Subanji et al., 2021). Such conflict arises when a discrepancy exists between students' prior mathematical knowledge and the controversial problem they face, often resulting in cognitive impasse and subsequent reevaluation (Mueller & Yankelewitz, 2014; Rolka & Liljedahl, 2007).

At the initial level, SI's inability to identify the actual source of contradiction, despite recognising that the result was implausible, exemplifies what prior research describes as surface-level, intuitive reasoning that fails to provide valid justifications for the approaches (Muniri & Musrikah, 2024) and do not arrive at the expected conclusions (El Walida & Sa'dijah, 2022). This limitation is likely linked to insufficient experience with proof tasks that require critical evaluation of assumptions (Chamberlain & Vidakovic, 2021; Song et al., 2017). Without mastery of formal definitions such as recursive sequence structure and the conditions for limit separation (Beynon & Zollman, 2015; Kristanto et al., 2019; Swinyard &

Larsen, 2012)Students cannot translate intuitive suspicion into a logically grounded argument.

At the exploration level, SE's reasoning demonstrates a meaningful but incomplete shift toward analytical thinking. The correct diagnosis of the convergence assumption, combined with a structured induction attempt, reflects the exploratory initiatives characteristic of this level (Hačatrjana & Namsone, 2024). However, SE's failure to recognise that a circular substitution invalidated the inductive step — and the subsequent application of the Tail Limit Theorem to an already-flawed proof — illustrates how reasoning at this level remains fragmented without strong metacognitive regulation (Muniri & Musrikah, 2024). SE produces biased conclusions not from a lack of relevant knowledge, but from an inability to integrate that knowledge coherently: the correct identification of the convergence problem coexists with an invalid inductive argument, revealing that students at this level draw from multiple knowledge sources without the regulatory capacity to verify their internal consistency (El Walida et al., 2024).

At the level of clarification, SK's approach is distinguished not by superior knowledge alone but by a qualitatively different relationship between exploration and formal justification. The deliberate use of MS Excel before committing to a deductive argument reflects stable coordination among empirical observation, formal logic, and critical reflection, a coherence absent at lower levels (Taylor, 2017; Muniri & Musrikah, 2024). This metacognitive sequencing, conjecture first, proof second, is precisely what is absent in SE's work and absent in SI's reasoning process. The ability to construct a proof by contradiction from scratch, rather than attempting to repair the flawed proof in the problem, further demonstrates the structured reasoning that defines clarification-level performance (Galiç et al., 2025; Hamami & Morris, 2024). Crucially, SK's approach shows that success at this level is not incidental but follows from deliberate verification of foundational assumptions, specifically, establishing unboundedness before invoking convergence, and systematic reflection on the logical validity of each step.

Taken together, the progression across levels indicates that the primary barrier in proof construction is not computational error but the inability to bridge intuitive suspicion with formal definitions — a gap that widens as task demands increase. The findings also highlight that what separates exploration from clarification is less about knowledge quantity and more about the capacity to verify foundational prerequisites before applying proof strategies and to reflect critically on the validity of intermediate conclusions.

CONCLUSION

This study confirms and extends the three-level controversial reasoning framework of Subanji et al. (2021) within the specific context of Real Analysis proof construction. Students at the initial level recognise logical implausibility but offer conceptually irrelevant justifications and no substantive attempt at proof, reflecting the absence of the prerequisite mastery needed to translate intuitive suspicion into a formal argument. At the exploration level, students correctly identify the source of contradiction and initiate structured proof strategies, yet fail to integrate foundational prerequisites — particularly convergence verification — due to insufficient metacognitive regulation, resulting in fragmented reasoning and biased conclusions. At the clarification level, students exhibit a qualitatively distinct mode of reasoning characterised by empirical-to-deductive sequencing, deliberate verification of assumptions, and coherent proof by contradiction. The primary barrier across levels is not computational error, but the inability to bridge intuitive suspicion with formal definitions. Notably, the distribution of students across levels — initial (n = 15, 37.5%), exploration (n = 24, 60%), and clarification (n = 1, 2.5%) — is itself a substantive finding: the near-absence of clarification-level performance is consistent with prior literature, and suggests that clarification-level reasoning represents a genuinely rare cognitive achievement in this population rather than a sampling artifact.

The finding that 97.5% of students are below the clarification level indicates that conventional Real Analysis instruction is insufficient to support the transition to rigorous proof-based reasoning. Instruction should explicitly integrate controversial problems to make contradiction productive: for initial-level students, through guided questioning that foregrounds the gap between intuition and formal definition; for exploration-level students, the majority, through activities that develop metacognitive habits of verifying prerequisites before applying theorems. At the curriculum level, the near-absence of clarification-level performance suggests that scaffolding the exploration-to-clarification transition requires deliberate and sustained instructional design, not incidental exposure to proof tasks.

Several limitations bound the interpretive scope of this study. Data were collected from a single institution, restricting transferability. The single-case-per-level design, while consistent with the study's exploratory aims, limits confirmation of intra-level patterns, particularly at the clarification level, where a single case cannot establish generalizability. Whether the extreme distributional skewness observed here is reproducible across institutional contexts remains an open empirical question. Future research should employ multi-site designs with larger sample sizes, longitudinal tracking to observe the development of reasoning over time, and intervention studies testing pedagogical approaches that specifically target the metacognitive gap between exploration and clarification.

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