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PROFILE OF PROSPECTIVE TEACHERS' ERRORS IN CONTRUCTING MATHEMATICS PROOF USING COUNTEREXAMPLES

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Abstract

Problems often encountered in mathematics are problems to prove and problems to find. Proving ability is very important, but it is still a weakness for most Mathematics Education students at Widya Mandala Catholic University Surabaya. This study describes the error profile of prospective teacher students in constructing mathematical proofs by using counterexamples and the factors suspected to be the cause of the error. This is exploratory-descriptive qualitative research, with six subjects of PSDKU Mathematics Education UKWMS who have taken Mathematical Logic and Set Theory. Data was collected using a test technique, the Mathematical Proof Ability Test, which was a subjective test consisting of 5 questions. The results of student work are then analyzed to describe the errors made by students in constructing mathematical proofs using counterexamples. The results showed that the errors made by students were: (1) errors in using the method of proof; (2) errors in using fundamental mathematical theorems; (3) errors in understanding the problem. The factors that are suspected to be the cause of student errors are: (1) students have a poor understanding of mathematical concepts; (2) students do not master the method of proof needed as a guide in carrying out mathematical proofs; (3) students do not have enough practice so they lack experience in constructing mathematical proofs by using counterexamples.
Keywords: Error Profile, Constructing Mathematical Proofs, Counterexample

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INTRODUCTION

One of the characteristics of mathematics is that it is deductive axiomatically. Mathematics is arranged hierarchically or in stages, starting with an agreement that can be seen from various axioms and definitions, then derived statements (theorems, propositions) whose truth must be proven before they can be used (Juandi, 2008). With these characteristics, it is not surprising that in mathematics, there is a very important ability, namely the ability to do proof. The ability to prove a mathematical statement or theorem is one of the essential abilities that must be possessed by students who study mathematics. This statement is based on the fact that a large part of college mathematics is a matter of proving. This probing activity teaches students to think critically and systematically, organize reasoning, and be creative. The ability to prove is also one of the higher-order thinking skills, which cannot be denied that this ability is indispensable in entering the era of the industrial revolution 4.0.

The ability to prove a mathematical statement since the 2013 curriculum was implemented has also become one of the topics discussed at the high school level. This shows that mathematical proof is a very important topic, so this topic needs to be studied early. Thus, mathematics teachers are also required to have good proving skills. Although proving theorems is a very important skill and indispensable for learning mathematics, unfortunately, this ability is a weakness for most PSDKU Mathematics Education students at Widya Mandala Catholic University, Surabaya. This weakness is obvious when students are asked to solve proof problems.

PSDKU Mathematics Education UKWMS is a formal educational institution that produces mathematics teachers in secondary schools and is responsible for producing high-quality secondary school mathematics teachers, one of which indicators have the ability related to mathematical proof. Therefore, considering that secondary school mathematics teachers must have reliable mathematical proof skills, graduates of the UKWMS Mathematics Education PSDKU must also be equipped with mathematical proof abilities.

From the description above, it can be felt that the ability to do mathematical proofs is very important in learning mathematics. However, it cannot be denied that most students consider mathematical proof difficult. The researcher felt students' difficulty in proving mathematics when giving lectures on Introduction to Basic Mathematics (now Mathematical Logic and Set Theory) and Real Analysis.

One of the factors causing the difficulties of the UKWMS Mathematics Education PSDKU students in carrying out mathematical proofs is their lack of understanding of mathematical proof methods (Yohanes, 2022). Students are often not precise in choosing the method of proof in carrying out mathematical proofs. In addition, many students still think that mathematical proofs must be general. In mathematics, you cannot prove by example. Students' perceptions like this make it difficult to prove the truth of a statement using existential quantifiers, which is actually enough to be proven by using an example.

Considering that mathematics teachers must be able to do reliable mathematical proofs, graduates of the UKWMS Mathematics PSDKU must also be equipped to do mathematical proofs. As a first step, the researchers conducted research on: The profile of Student Errors in Proving Mathematics, considering that research on the errors of PSDKU students in Mathematics Education UKWMS has not been done much. By knowing students' mistakes in mathematical proofs, researchers can find out the weaknesses that occur when students do mathematical proofs to find ways to overcome them. The results of this study can also be used as a database to conduct further research on efforts to improve students' mathematical proofing skills so that the Mathematics Education PSDKU has a map of strengths and weaknesses regarding students' ability to perform mathematical proofs.

Based on the background of the problem above, the problems to be investigated in this research are 1) what is the error profile of UKWMS Mathematics Education PSDKU students constructing mathematical proofs using examples of deniers? 2) What factors are thought to cause student errors in proving mathematics?

According to (Polya, 1981), there are two types of mathematical problems, namely problems to find and problems to prove. Since mathematics has axiomatic deductive nature, every statement must be proven true based on hypotheses or statements that have been proven true. Therefore, the proof is a very important thing in mathematics. The ability to prove is an ability that must be mastered well for people who study mathematics.

One of the main problems in mathematics is investigating the truth of a statement; for example, the statement *p* is taken: "Every member of set *A* is a member of set *B*". To show that the statement p is "true", proof must be offered that A is contained in B. Meanwhile, to show that the statement *p* is "false", it must be shown that there is a member of set *A* that is not a member of set *B*. In other words, an example must be constructed that "denies" the truth of statement *p*. Such an example is an "example of denial" for a *p* statement (Gogovska, 2015).

The statement "Every member of set *A* is a member of set *B*" is an implication. If the universal set is the set *x*, then the above statement can be presented as follows: $(\forall x \in X)(x \in A \Rightarrow x \in B)$. To show that the above statement is false, we must indicate the existence of an element $y \in X$ with $y \in A$ but $y \notin B$. So $(\exists y \in X)(y \in A \text{ dan } y \notin B)$. Sometimes to look for such a *y* element, for example, to make an example of a refutation for a statement, is not an easy job; it requires deep and broad thinking, so it is an interesting and stimulating problem to solve.

One of the functions of an example a disclaimer is to prove that a statement is not generally valid or to refute an assumption that is still doubtful. If an example can be built to refute the statement or allegation, it means that it has been proven that the statement or allegation is wrong. Shorser (2012) say that the example of a disclaimer is an example to show that a statement is not always true, so it is enough to give an example.

Example:

In 1640, Fermat conjectured that a natural number of the form $2^{2^m}+1$, with $m \in N$ s a prime number. By Euler, Fermat's conjecture is invalidated by an example of denial, namely $m = 5$, and the form is divisible by 641.

$$2^{2^5}+1 = 2^{32}+1 = 4294967297 = 641 \times 6700417$$

In honour of his work, prime numbers in the form $2^{2^m}+1$, with $m \in N$ are called Fermat numbers (Cindy, 2010); (Klymchuk, 2008).

Bartle dan Sherbert (Bartle & Sherbert, 2010) also provide examples of the benefits of the denier example: If $P(n) = n^2 - n + 41$, with $n \in N$, is each $n \in N$, $P(n)$ a prime number?

Apparently "not". Although for n = 1, 2, 3, ... , 40, P(n) is a prime number, but for n = 41, P(n) is not a prime number, because P(41) is divisible by 41.

METHOD

This type of research is descriptive-explorative qualitative research. It seeks to explain or describe the research findings and seek answers (exploration) on students' abilities in constructing mathematical proofs along with their weaknesses and strengths.

The subjects of this study were 6 PSDKU Mathematics Education students who had taken high school mathematics courses. Therefore, the mathematical proof material used in this study is high school mathematics.

The data collection technique used in this research is a test technique. For example, describing the profile of students' abilities in constructing mathematical proofs using examples of denial is done by analyzing and interpreting the steps or methods students use in constructing mathematical proofs. Researchers also use interview techniques if researchers have difficulty interpreting the steps or methods used by students.

The research instrument used in this study was the Mathematical Proofing Ability Test (TKPM). The TKPM material is focused on high school mathematics. TKPM is a description of 5 questions which ask students to answer wholly and systematically.

The results of student work were analyzed qualitatively to find out the error profile of the UKWMS Mathematics Education PSDKU students in constructing mathematical proofs by using examples of deniers who seek to:

a. Describe the student error profile in performing mathematical proofs by using an example of denial.
b. Exploring the factors that are thought to cause the UKWMS Mathematics

Education PSDKU students to make mistakes in doing mathematical proofs by using examples of denial.

RESULT AND DISCUSSION

The following is a profile of the errors of the UKWMS Mathematics Education PSDKU students in carrying out mathematical proofs using an example of denial. This paper presented three problems that most students experience errors.

Problem Number 1:

Find out if $x > y$, then $\frac{x}{y} > 1$, applies to every *x* and *y* real number. Prove your answer.

Answer:

Statement: If $x > y$, then $\frac{x}{y} > 1$, valid for every *x* and *y* real number is a false statement.

Proof:

It is proved by using an example of a disclaimer:

Choose $x = 5$ and $y = -10$, then $x > y$, but $\frac{x}{y} = \frac{5}{-10} = -\frac{1}{2} < 1$.

Since there are $x = 5$ and $y = -10$, so that $x > y$ but $\frac{x}{y} < 1$, then the statement: If $x > y$, then $\frac{x}{y} > 1$, for every real number *x* and *y* is a false statement.

Result of Problem Analysis Number 1:

For problem number 1, from six students, there were three students (M2, M4, M6) who answered incorrectly and three students (M1, M3, M5) who answered correctly. Students M2, M4, and M6 think the statement in problem number 1 is true. However, in the proving process, it appears that students make mistakes in using basic mathematical operations and performing mathematical manipulations. In constructing the proof, M2 students made many mistakes in performing mathematical manipulations, so the conclusions drawn were wrong. Meanwhile, the mistake made by M4 students was to prove the truth of a generally accepted statement using an example (inductively). At the same time, a generally accepted view should be proven deductively. Then the mistake made by M6 students is to use the property of the inequality that is not true.

The following results of the work of M2, M4, and M6 students are presented.

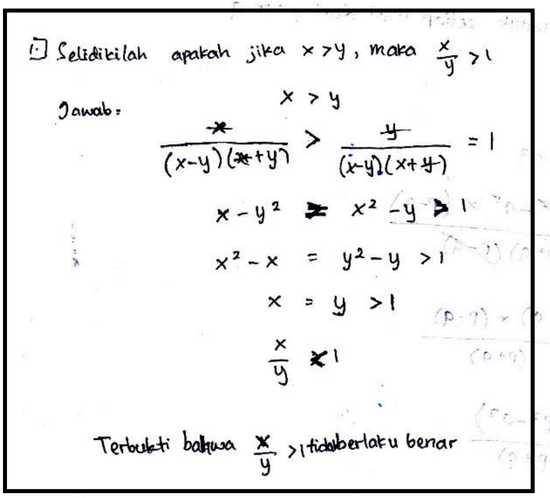


Figure 1. M2 Student Work for Problem Number 1

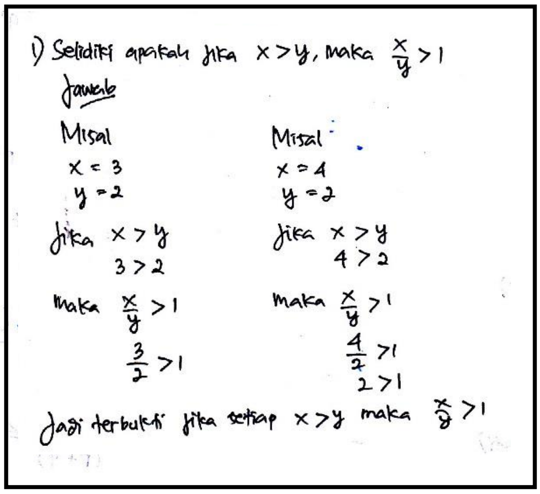


Figure 2. M4 Student Work for Problem Number 1

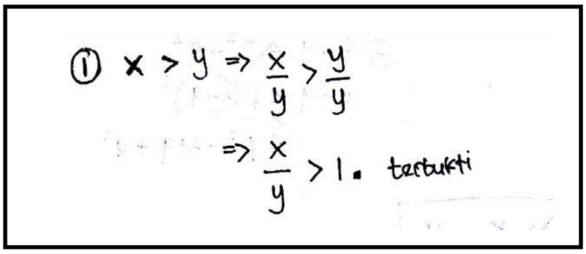


Figure 3. M6 Student Work for Problem Number 1

Problem Number 2:

Find out whether $p - q \leq p^2 - q^2$ holds for every p and q real number. Prove your answer.

Answer:

$p - q \leq p^2 - q^2$ valid for every p and q real numbers is a false statement.

Proof:

It is proved by using an example of a disclaimer:

Choose $p = 6$ and $q = -8$,

Then we get : $p - q = 6 - (-8) = 6 + 8 = 14$

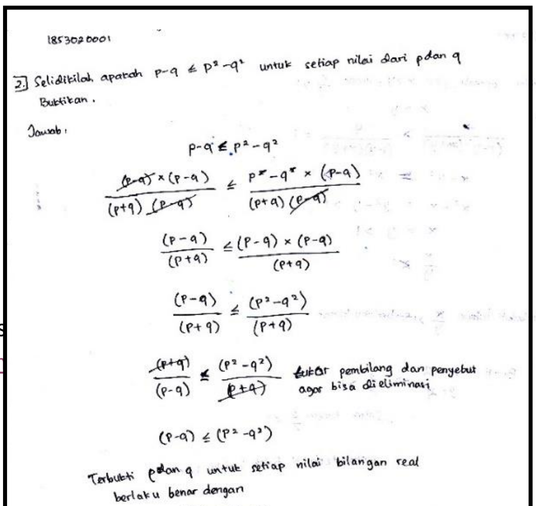
$$p^2 - q^2 = \left(6^2 - (-8)^2\right) = 36 - 64 = -28$$

Since there are $p = 6$ and $q = -8$, so that $p - q \geq p^2 - q^2$, it means that the statement: If $p - q \leq p^2 - q^2$, for every real number *x* and *y* is a false statement.

Result of Problem Analysis Number 2:

For problem number 2, of the six students, there were three students (M2, M4, M5) who answered incorrectly and two students (M1, M3) who answered correctly, while one student (M6) did not answer. Students M2, M4, and M5 think that the statement in problem number 2 is true. In the proving process, it appears that students make mistakes in mathematical operations and performing mathematical manipulations. M2 and M4 students do not understand which one is known as the starting point and which one must be proven. Students M2 and M4 start from the inequality $p - q \leq p^2 - q^2$ which should be something to prove. In the proving process, M2 and M4 made mistakes in manipulating mathematical forms or using unclear steps, so they seemed to be messing around and returning to the initial inequality, which M2 and M4 considered as being proven. M2 and M4 do not understand what is known and what must be proven.

The following are the results of the work of M2 and M4 students for problem 2.



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2.) Selidikilah apakah $p-q \leq p^2-q^2$ untuk setiap nilai dari p dan q
Buktikan.

Jawab :

$$p-q \leq p^2-q^2$$

$$\frac{\cancel{(p-q)} \times (p-q)}{(p+q)\cancel{(p-q)}} \leq \frac{p^2-q^2 \times (p-q)}{(p+q)\cancel{(p-q)}}$$

$$\frac{(p-q)}{(p+q)} \leq \frac{(p-q) \times (p-q)}{(p+q)}$$

$$\frac{(p-q)}{(p+q)} \leq \frac{(p^2-q^2)}{(p+q)}$$

$$\frac{\cancel{(p+q)}}{(p-q)} \leq \frac{(p^2-q^2)}{\cancel{(p+q)}}$$ tukar pembilang dan penyebut agar bisa dieliminasi

$$(p-q) \leq (p^2-q^2)$$

Terbukti p dan q untuk setiap nilai bilangan real berlaku benar dengan

Figure 4. M2 Student Work for Problem Number 2

The mistakes made by M5 students were errors using the basic nature of mathematics. For example, students argue that for p positive real numbers, then $0 \leq p \leq p^2$. This opinion is not true. This opinion only applies $p \geq 1$ and does not apply $0 \leq p \leq 1$. The evidence made by M5 students is also incomplete because it has only proven that the values of *p* and *q* are positive real numbers; what about the values of p and q, which are negative real numbers?

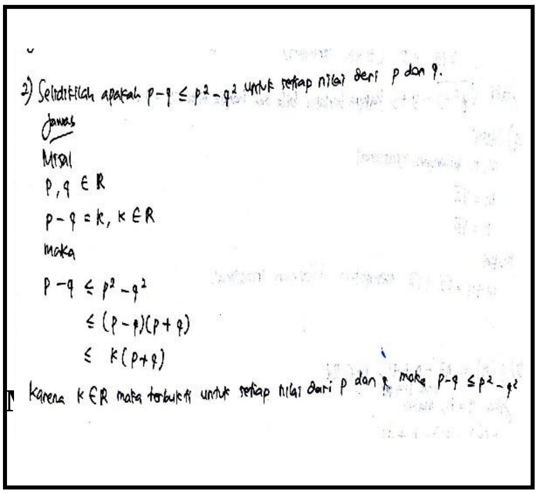

2) Selidikilah apakah $p-q \leq p^2-q^2$ untuk setiap nilai dari p dan q.
Jawab
Misal
$p, q \in R$
$p-q = k, k \in R$
maka
$p-q \leq p^2-q^2$
$\leq (p-q)(p+q)$
$\leq k(p+q)$
Karena $k \in R$ maka terbukti untuk setiap nilai dari p dan q maka $p-q \leq p^2-q^2$

Figure 5. M4 Student Work for Problem Number 2

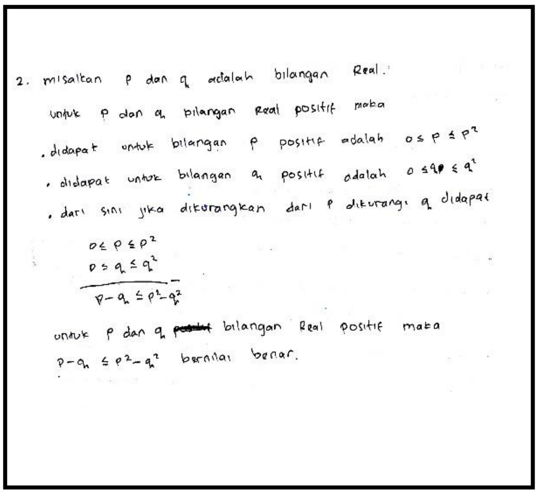

2. misalkan p dan q adalah bilangan Real.
untuk p dan q bilangan Real positif maka
. didapat untuk bilangan p positif adalah $0 \leq p \leq p^2$
. didapat untuk bilangan q positif adalah $0 \leq q \leq q^2$
. dari sini jika dikurangkan dari p dikurangi q didapat
$0 \leq p \leq p^2$
$0 \leq q \leq q^2$
$p-q \leq p^2-q^2$
untuk p dan q ~~positif~~ bilangan Real positif maka
$p-q \leq p^2-q^2$ bernilai benar.

Figure 6. M5 Student Work for Problem Number 2

Find out whether $p(n) = n^2 - n + 41$ is a prime number for every n natural numbers.

Prove your answer.

Answer:

$p(n) = n^2 - n + 41$ is a prime number for every n natural numbers is a false statement.

Proof:

It is proved by using an example of a disclaimer:

Choose: n = 41.

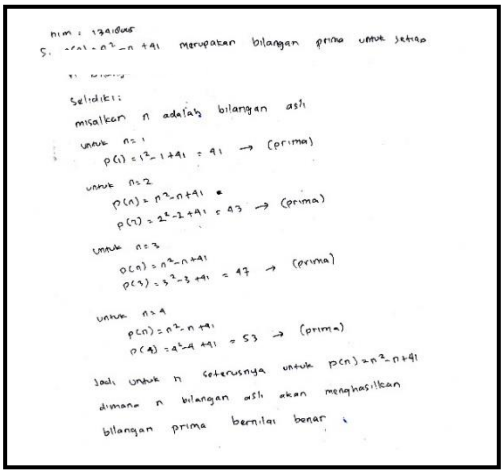
$p(41) = 41^2 - 41 + 41$ not prime number, because p(41) is divisible by 41.

Thus, $p(n) = n^2 - n + 41$ is a prime number for every n natural numbers is a false statement.

Result of Problem Analysis Number 3:

To prove problem number 3, from six students. Four students (M2, M4, M5, M6) answered incorrectly, and two (M1, M3) answered correctly. Students M2, M4, M5, and M6 think that the statement in problem number 3 is true. Student errors M2, M4, M5, and M6, are errors using the proof method. They prove the statement's truth that applies to every natural number by using examples, which should be proven deductively.

The following is an example of student work for problem number 3.



5. $p(n) = n^2 - n + 41$ merupakan bilangan prima untuk setiap

Selidiki:

misalkan n adalah bilangan asli

untuk n = 1

$p(1) = 1^2 - 1 + 41 = 41 \rightarrow$ (prima)

untuk n = 2

$p(n) = n^2 - n + 41$

$p(2) = 2^2 - 2 + 41 = 43 \rightarrow$ (prima)

untuk n = 3

$p(n) = n^2 - n + 41$

$p(3) = 3^2 - 3 + 41 = 47 \rightarrow$ (prima)

untuk n = 4

$p(n) = n^2 - n + 41$

$p(4) = 4^2 - 4 + 41 = 53 \rightarrow$ (prima)

Jadi untuk n seterusnya untuk $p(n) = n^2 - n + 41$ dimana n bilangan asli akan menghasilkan bilangan prima bernilai benar.

Figure 7. M5 Student Work for Problem Number 3

From the results of data analysis on student error profiles in carrying out mathematical proofs using. Examples of denial, it can be stated that factors that can be suspected as causes of these errors, namely: (1) students are less trained/trained, so they are less experienced in doing mathematical proofs; (2) students do not master the method of proof that is needed as a guide in carrying out mathematical proofs; (3) Students do not master the basic properties of mathematics.

Based on the analysis of student errors in performing mathematical proofs by using examples of denial, it appears that students still make mistakes. Errors made by students include (1) errors in using the method of proof; (2) errors in using basic mathematical theorems; (3) errors in understanding the problem so that students are often wrong in determining what is known and which must be proven. The results of this analysis are in line with research (Mujib, 2019); (Sari et al., 2017); (Watson & Mason, 2005).

Although this paper has described the factors that are suspected to be the cause of student weaknesses in carrying out mathematical proofs, namely: (1) students' poor understanding of mathematical concepts; (2) students who do not master the method of proof needed as a guide in carrying out mathematical proofs; (3) students are poorly trained so that they lack experience in constructing mathematical proofs by using examples of denial, but researchers need to emphasize that the causal factors described in this paper are only conjectures. To find out the real causative factors, further research is still needed. However, students' weaknesses in carrying out mathematical proofs and the factors that cause these weaknesses can be input for the UKWMS Mathematics Education PSDKU in general and the lecturers who teach mathematics courses whose course content contains mathematical proofs that they can be searched. Solutions, as well as students, improve their ability to do mathematical proofs.

CONCLUSION

The following conclusions can be drawn based on the data analysis and discussion described above.

a. The results of the work of 6 students in carrying out mathematical proofs using examples of denial still make many mistakes. Errors made by students include: (1) errors in using the method of proof; (2) errors using basic mathematical theorems; (3) errors in using the information provided in the problem, for example students are still often wrong in determining what is known and which must be proven.

b. The factors that are suspected to be the cause of students' weaknesses in performing mathematical proofs are: (1) students' poor understanding of mathematical concepts; (2) students do not master the method of proof needed as a guide in carrying out mathematical proofs; (3) students are not trained, so they lack experience in constructing mathematical proofs by using examples of denial.

Some suggestions are expected to be useful in overcoming the existing weaknesses.

a. Provide more and more frequent experience for students to do mathematical proofs. For highly deductive subjects, the tasks given to students should focus on proving mathematical theorems to practice their ability to do mathematical proofs.

b. Before learning mathematical proof, students first need to learn to understand the steps of an existing mathematical proof.

c. In providing examples of mathematical proofs, lecturers should provide complete and systematic proof steps so that students understand the process of mathematical proofing.

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